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Notes on R-charged black holes near criticality and gauge theory

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Abstract: After reviewing the thermodynamics and critical phenomena associated with AdS black holes carrying multiple R-charges in various dimensions, we carry out a Bragg-Williams like analysis of the systems around its critical points. This leads us to propose an effective potential governing the equilibrium properties of the boundary gauge theory. We also study certain non-equilibrium phenomena associated with these gauge theories. In particular, we compute the conductivities and diffusion coefficients for theories with multiple R-charges in four, three and six dimensions.

Keywords: Gauge-gravity correspondence, Black Holes in String Theory, AdS-CFT Correspondence, D-branes

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1 Introduction

Recently there appear stimulating studies concerning spinning near-extremal D3 branes [1–9]. There are at least two reasons that motivate these activities. The more important one is, perhaps, the fact that this provides a simple example in the framework of gauge-gravity duality where different aspects can be studied in presence of non-zero charge as well as chemical potential. Indeed, the ten dimensional gravitational background corresponding to near extremal spinning D3 brane provides gravity dual of a four dimensional $\mathcal{N} = 4$ finite temparature super Yang-Mills theory. The number of independent angular momenta of the spinning D3 brane is the rank of the isometry group SO(6) of the space tranverse to the brane, which upon dimensional reduction corresponds to charged black hole with three U(1) charges. In the gauge theory, this isometry group represents the R-symmetry group. Three independent chemical potentials can therfore be introduced which couple to the three independent U(1) charges. By exploiting the gauge-gravity correspondence, several recent works explored the behaviour of such gauge theories at strong coupling in its simplest setting. These are the cases where only one of the chemical potentials is non-zero.

Another reason that makes the study of rotating D3 brane interesting is the fact that spinning D3 brane has an upper critical limit of angular momentum beyond which thermodynamic instability sets in. Though the final configuration to which this instability leads is yet to be understood, it is conjectured that when angular momentum reaches this critical value, the system undergoes a phase transition. A careful inspection of the bahaviour of various thermodynamic quantities suggests that it is a second order phase transition. In particular, near this critical point, divergence of correlation length shows up in the behaviour of thermodynamic quantities such as specific heat and non-equilibrium quantities such as, transport coefficients. One expects that, in the vicinity of the second order phase transition, certain physical properties of the system can be extracted without the knowledge of their microscopic behaviours.

With these motivations, in this paper, we study various aspects of the R-charged black holes near its critical point. So far these studies were carried out for the simplest variant of this model, namely, D3 branes with a single R-charge. The purpose of this article is to extend these stuides to its full generality. This leads to more intricate phase structures of the systems along with interesting surprises. So far the equlibrium thermodynamics is concerned, we carry out a Bragg-Williams like analysis in the bulk. More specifically, by identifying the horizon radius as the order parameter, we write down the Bragg-Williams [10] potential for the R-charged black hole. We also extend our analysis for charged black holes in four and seven dimensions. They appear as the near horizon limits of non-extremal spinning M2 and M5 branes. Subsequently, by exploiting the gauge-gravity duality, we illustrate with an explicit example, how this leads to a proposal of effective potential for the gauge theory. As far as the non-equilibrium phenomena are concerned, we study the transport coefficients of the gauge thery duals using gauge-gravity correspondence. Besides giving a general precription to solve perturbed gravity equations for multiple R-charged black holes in four, five and seven dimensions, we explicitly compute the diffusion coefficients for all these cases using the Green-Kubo formula.

The paper is structured as follows. In the next section, we reconsider various R-charged black holes, in the grand canonical enseble, with a focus on their thermodynamic properties [2]. On one hand, this section allows us to set our notations and conventions and on the other hand, it helps us to scan the complicated phase structure associated with the black holes with multiple R-charges in various dimensions. Next, in section 3, we consider the black holes near their critical temperatures. By identifying horizon radius as the order parameter, we construct the Bragg-Williams like potentials for the black holes. This further leads us to propose an effective potential for the boundary gauge theory with charge density as the order parameter. We illustarate this explicitly for the four dimensional gauge theory with a single non-zero chemical potential. Section 4 is devoted to a study of certain non-equilibrium phenomena associated with these bulk geometries and its imprint on the boundary theories. In particular, we compute transport coefficients using Green-Kubo formula. It turns out that for multiple R-charges, the calculations are quite involved. We, therofore, in the main text restrict ourselves to the evaluation of the R-charge conductivity for black holes when two charges are non-zero deferring rest of the cases to the appendix. The appendix also includes an algorithm to compute the conductivities in various dimensions and for multiple charges. Finally, we conclude the paper in section 5 with a discussion on our results.

2 Black holes in STU model

In this section we review thermodynamics of the R-charged black hole in five, four and seven dimensions. We start with the black holes in five dimensions. These are the solutions of equations of motion of $D = 5$, $\mathcal{N} = 2$ gauged supergravity obtained by compactification of ten dimensional IIB supergravity on S^5. These are known as STU black holes and were found in [11–13]. The effective action is given by

$$S = \frac{1}{16\pi G}\int \sqrt{-g}d^5x\bigg(R + \frac{2}{l^2}\mathcal{V} + \frac{1}{2}G_{ij}F^i_{\mu\nu}F^j_{\mu\nu} - G_{ij}\partial_\mu X^i\partial^\mu X^j + \frac{24}{\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma\lambda}\epsilon_{ijk}F^i_{\mu\nu}F^{\rho\sigma j}A^k_\lambda\bigg), \tag{2.1}$$

where l represents the scale associated with the cosmological constant. In addition to the metric, we have three scalar fields X^i, $i = 1, 2, 3$ which are constrained through the relation $X^1X^2X^3 = 1$. $\mathcal{V}$ is a potential involving the scalar fields given by

$$\mathcal{V} = 2\sum_1^3 \frac{1}{X^i}. \tag{2.2}$$

G_{ij} represents metric of the moduli space parametrized by the scalars and is given by

$$G_{ij} = \frac{1}{2}\text{diag}\Big[(X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\Big]. \tag{2.3}$$

$F^i_{\mu\nu}$ are the field strengths associated with three abelian gauge fields A^i corresponding to three U(1) in the cartan of the R-symmetry group.

As shown in [11], this effective action (2.1) admits asymptotically AdS black hole solutions with three U(1) charges. These solutions can be written down as

$$ds^2 = -\mathcal{H}^{-\frac{2}{3}}\frac{r^2}{l^2}f dt^2 + \mathcal{H}^{\frac{1}{3}}\frac{l^2}{fr^2}dr^2 + \mathcal{H}^{\frac{1}{3}}\frac{r^2}{l^2}(dx^2 + dy^2 + dz^2), \tag{2.4}$$

where

$$f = \left(1 + \frac{q_1}{r^2}\right)\left(1 + \frac{q_2}{r^2}\right)\left(1 + \frac{q_3}{r^2}\right) - \frac{r_0^4}{r^4}, \tag{2.5}$$

and the harmonic functions are given by

$$\mathcal{H} = H_1H_2H_3 = \left(1 + \frac{q_1}{r^2}\right)\left(1 + \frac{q_2}{r^2}\right)\left(1 + \frac{q_3}{r^2}\right). \tag{2.6}$$

In the above solutions, the parameter r_0 is related to the mass or energy density of the black hole while q_1, q_2, q_3 are related to the charges of the black hole.

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2.1 Black hole thermodynamics and instabilities in five dimensions

In this subsection we briefly describe the thermodynamics and the critical phenomena associated with the five dimensional R-charged black hole given by (2.4), (2.5), (2.6). From now, instead of using r_0 as a parameter, we will use the radius of the outer horizon r_+ to parametrize the solution. r_+ can be obtained from the largest root of the equation $f(r_+) = 0$, where $f(r)$ is given in (2.5). Furthermore, we will scale all the dimensionful parameters with AdS length scale l e.g. we introduce the dimensionless parameter $\bar{r} = r_+/l$ and $\bar{T} = lT$ for horizon radius and Hawking temperature respectively. Similarly the dimensionless charge parameters will be denoted by $\bar{q}_i = q_i/l^2$. In terms of these scaled parameters, horizon area becomes proportional to

$$A = \sqrt{(\bar{r}^2 + \bar{q}_1)(\bar{r}^2 + \bar{q}_2)(\bar{r}^2 + \bar{q}_3)}. \tag{2.7}$$

In addition, we will set Newton's constant $G = 1$. G and l will be resurrected whenever required. In what follows we will consider the system in the grand canonical ensemble with temparature T and chemical potentials μ_i, $i = 1, 2, 3$ as external parameters.

The scaled Hawking temperature turns out to be

$$\bar{T} = \frac{2\bar{r}^6 + \bar{r}^4(\bar{q}_1 + \bar{q}_2 + \bar{q}_3) - \bar{q}_1\bar{q}_2\bar{q}_3}{2\pi\bar{r}^2 A}. \tag{2.8}$$

Other thermodynamic quantities such as entropy, energy and pressure are expressed as

$$\bar{S} = \frac{1}{4}A\,, \qquad \bar{E} = \frac{3}{16\pi\bar{r}^2}A^2\,, \qquad \bar{P} = \frac{1}{16\pi\bar{r}^2}A^2\,, \tag{2.9}$$

where the above quantities are rendered dimensionless by multiplying appropriate power of l. The components of the charge density and chemical potential are given by

$$\bar{\rho}_i = \frac{\sqrt{2\bar{q}_i}}{16\pi\bar{r}}A, \quad \bar{\mu}_i = \frac{\sqrt{2\bar{q}_i}}{\bar{r}(\bar{r}^2 + \bar{q}_i)}A. \tag{2.10}$$

Since we are analysing black holes with flat horizon, the horizon has infinite volume. Consequently, only the themodynamic densities are finite. What we have written above actually represent those thermodynamic densities.

Unless charges satisfy certain constraints, these black holes undergo a local instability [2, 3, 14]. While at high temperature, black holes remain stable, once we reduce the temperature down to a critical value, the specific heat and suceptibility diverge. In order to see this, let us compute those quantities. The specific heat associated with the black holes has the following form

$$\begin{aligned}\bar{C} = \left(\bar{T}\frac{\partial\bar{S}}{\partial\bar{T}}\right)_{\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3} =&(2\bar{r}^6 + \bar{r}^4(\bar{q}_1 + \bar{q}_2 + \bar{q}_3) - \bar{q}_1\bar{q}_2\bar{q}_3)\\ &\times\frac{3\bar{r}^6 - \bar{r}^4(\bar{q}_1+\bar{q}_2+\bar{q}_3) - \bar{r}^2(\bar{q}_1\bar{q}_2+\bar{q}_2\bar{q}_3+\bar{q}_3\bar{q}_1)+3\bar{q}_1\bar{q}_2\bar{q}_3}{4A(2\bar{r}^6 - \bar{r}^4(\bar{q}_1 + \bar{q}_2 + \bar{q}_3) + \bar{q}_1\bar{q}_2\bar{q}_3)}.\end{aligned} \tag{2.11}$$

The susceptibility, for the present case is a 3×3 symmetric matrix, defined as

$$\bar{\chi}_{ij} = \left(\frac{\partial\bar{\rho}_i}{\partial\bar{\mu}_j}\right)_{\bar{T},\bar{\mu}_k,\bar{\mu}_l}, \text{ with } \; k, l \neq j. \tag{2.12}$$

Two of its components are given by

$$\bar{\chi}_{11} = \frac{2\bar{r}^8 + 5\bar{r}^6\bar{q}_1 - \bar{r}^4\bar{q}_1^2 - \bar{r}^6\bar{q}_2 - \bar{r}^4\bar{q}_1\bar{q}_2 - \bar{r}^6\bar{q}_3 - \bar{r}^4\bar{q}_1\bar{q}_3 - 3\bar{r}^2\bar{q}_1\bar{q}_2\bar{q}_3 + \bar{q}_1^2\bar{q}_2\bar{q}_3}{16\pi(2\bar{r}^6 - \bar{r}^4(\bar{q}_1+\bar{q}_2+\bar{q}_3) + \bar{q}_1\bar{q}_2\bar{q}_3)},$$
$$\bar{\chi}_{12} = -\frac{\sqrt{\bar{q}_1\bar{q}_2}}{8\pi}\frac{\bar{r}^4\bar{q}_1 + \bar{r}^4\bar{q}_2 - \bar{r}^4\bar{q}_3 - \bar{q}_1\bar{q}_2\bar{q}_3}{2\bar{r}^6 - \bar{r}^4(\bar{q}_1+\bar{q}_2+\bar{q}_3) + \bar{q}_1\bar{q}_2\bar{q}_3}. \tag{2.13}$$

The other components can be obtained by cyclically permuting the indices (123). We now note that the specific heat and the suceptibility components diverge over the critical hypersurface

$$2\bar{r}^6 - \bar{r}^4(\bar{q}_1+\bar{q}_2+\bar{q}_3) + \bar{q}_1\bar{q}_2\bar{q}_3 = 0. \tag{2.14}$$

The thermodynamic quantities for a subclass of black holes where one or more charges are turned off can be easily obtained from the above by appropriate substitutions. Apart from some isolated cases, which we will discuss in the sequel, all these black holes show instabilities. The only difference is that hypersurfaces at which divergences occur are different from (2.14) and depend on the nature of the black hole. This is listed below for black holes with single, double or three charges:

$$\begin{aligned} 2\bar{r}^2 - \bar{q}_1 &= 0, && \text{for } \bar{q}_1 \neq 0, \bar{q}_2 = \bar{q}_3 = 0,\\ 2\bar{r}^2 - \bar{q}_1 - \bar{q}_2 &= 0, && \text{for } \bar{q}_1, \bar{q}_2 \neq 0, \bar{q}_3 = 0,\\ 2\bar{r}^4 - \bar{q}_1\bar{r}^2 - \bar{q}_1\bar{q}_2 &= 0, && \text{for } \bar{q}_1 \neq 0, \bar{q}_2 = \bar{q}_3 \neq 0. \end{aligned} \tag{2.15}$$

We also note that thermodynamic quantities are everywhere well behaved in the cases when charges satisfy certain restrictions. These include $\bar{q}_1 = \bar{q}_2$, $\bar{q}_3 = 0$ and $\bar{q}_1 = \bar{q}_2 = \bar{q}_3$. The second one is the standard Reissner-Nordstrom black hole in AdS space. Theromodynamic properties of these black holes were discussed in detail in [3]. The critical surfaces given in (2.15) can be expressed as relations between temperature and chemical potentials. For example, for one and two charge black holes, they are given respectively by

$$\bar{T} = \frac{\bar{\mu}_1}{\pi},$$
$$\bar{T} = \frac{\{\bar{\mu}_1^2 + \bar{\mu}_2^2 + (\bar{\mu}_1-\bar{\mu}_2)^{\frac{4}{3}}(\bar{\mu}_1+\bar{\mu}_2)^{\frac{2}{3}} + (\bar{\mu}_1-\bar{\mu}_2)^{\frac{2}{3}}(\bar{\mu}_1+\bar{\mu}_2)^{\frac{4}{3}}\}^{\frac{3}{2}}}{\pi\sqrt{4\bar{\mu}_1^2\bar{\mu}_2^2 + 3\{\bar{\mu}_1^2 + \bar{\mu}_2^2 + (\bar{\mu}_1-\bar{\mu}_2)^{\frac{4}{3}}(\bar{\mu}_1+\bar{\mu}_2)^{\frac{2}{3}} + (\bar{\mu}_1-\bar{\mu}_2)^{\frac{2}{3}}(\bar{\mu}_1+\bar{\mu}_2)^{\frac{4}{3}}\}^2}}. \tag{2.16}$$

For a generic three charged black hole, we have not been able to find the critical surface in terms of temperature and chemical potentials in a compact form. For a given set of $\bar{\mu}_i$, (2.16) gives us the critical temperature at which black hole instability arises. As we do not yet know the stable configuration below this temperature, in this paper we consider the black holes slightly *above* the critical temperature.

As one approaches the critical surface the correlation length diverges. This shows up in the divergences of thermodynamic quantities. Near the critical temperature, these black holes have universal features and their behaviour do not depend on details of the theory. These features are encoded in the critical exponents associated with behaviour of various thermodynamic quantities as we approach the critical surface.

We begin with the following four static critical exponents. Approaching the critical surface with fixed μ_i, one finds three critical exponents α, β, γ associated with the bahaviour of specific heat, charge densities and susceptibilities respectively.

$$\bar{C} \sim (\bar{T} - \bar{T}_c)^{-\alpha}, \bar{\rho} - \bar{\rho}_c \sim (\bar{T} - \bar{T}_c)^{\beta}, \bar{\chi} \sim (\bar{T} - \bar{T}_c)^{-\gamma}. \tag{2.17}$$

Since there are three U(1)'s, there are three charge densities and so the number of respective exponents β_i is also three. However, it turns out that they are all equal and one gets only a single β. Similarly, the susceptibility matrix χ_{ij} being symmetric and real, can be diagonalized as $\chi_{ij} = \chi_i \delta_{ij}$ and generically one would expect three γ's one for each of the components. But since all the three components are equal we get one γ only. On the other hand, exponent δ is obtained by approaching the critical line with a trajectory on which the temperature is constant. This is defined as

$$\bar{\rho} - \bar{\rho}_c \sim -(\bar{\mu} - \bar{\mu}_c)^{\frac{1}{\delta}}. \tag{2.18}$$

We find for all the black holes which show instability have the same set of critical exponents

$$(\alpha, \beta, \gamma, \delta) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2). \tag{2.19}$$

These critical exponents are not independent but are related through scaling relations. These critical exponents satisfy

$$\alpha + 2\beta + \gamma = 2, \quad \gamma = \beta(\delta - 1). \tag{2.20}$$

There are two other critical exponents ν and η associated with the behaviour of correlation length and correlation function near critical surface. If $G(\vec{r})$ is the correlator its behaviour near the critical surface is

$$\begin{aligned} &G(\vec{r}) \sim e^{-r/\xi} \qquad \text{at} \quad T \neq T_c, \quad \xi \sim \left(\frac{T - T_c}{T_c}\right)^{-\nu} \\ &G(\vec{r}) \sim r^{-d+2-\eta}, \quad \text{at} \quad T = T_c. \end{aligned} \tag{2.21}$$

Assuming additional scaling relations $\gamma = \nu(2 - \eta)$ and $2 - \alpha = \nu d$, to be valid we can determine ν and η to be equal to $\frac{1}{2}$ and 1 respectively. However, it is not clear a priory whether the scaling relations are valid. One can compute ν and η from correlation of scalar modes in the gravity theory. However, if we assume the scaling relations are valid then this will belong to class of B-model in classification given in [15].

As we will see in the next section, it is possible to construct a Bragg-William energy function which reproduces all these critical exponents near the second order phase transition surfaces. This function, in turn, will allow us to propose an effective potential for the boundary gauge theory via AdS/CFT correspondence.

2.2 Black hole thermodynamics and instabilities in four and seven dimensions

In this subsection, we briefly discuss the thermodynamics and instabilities of R-charged black holes in four and seven dimensions. As mentioned earlier, these configurations are obtained from spinning $M2$ and $M5$ branes of eleven dimension supergravity. First, let us focus on the black holes in four dimensions. They can carry atmost four independent charges arising form Cartans of SO(8). The black hole solutions are discussed in [12]. For our purpose, we require the thermodynamic quantities only. Our notation remains the same as in the last subsection except now there will be four charges q_i, $i = 1, 2, 3, 4$. For convenience, once again we introduce the scaled variables $\bar{r}$ and $\bar{q}_i$ and the horizon area is proportional to

$$A = \sqrt{(\bar{r} + \bar{q}_1^2)(\bar{r} + \bar{q}_2^2)(\bar{r} + \bar{q}_3^2)(\bar{r} + \bar{q}_4^2)}. \tag{2.22}$$

The temperature is given by

$$\bar{T} = \left(-\frac{1}{\bar{r}} + \sum_{j=1,4} \frac{1}{\bar{r} + \bar{q}_j^2} \right) A. \tag{2.23}$$

The entropy, energy and pressure are given by

$$\bar{S} = \frac{1}{4} A\,, \qquad \bar{E} = \frac{1}{8\pi\bar{r}} A^2\,, \qquad \bar{P} = \frac{1}{36\pi\bar{r}} A^2\,. \tag{2.24}$$

. The components of charge and chemical potentials are

$$\bar{\rho}_i = \frac{1}{32\pi} \sqrt{\frac{2\bar{q}_i^2}{\bar{r}}} A\,, \qquad \bar{\mu}_i = \sqrt{\frac{2\bar{q}_i^2}{\bar{r}(\bar{r} + \bar{q}_i^2)^2}} A\,. \tag{2.25}$$

The specific heat is given by

$$\bar{C} = \left(\bar{T} \frac{\partial \bar{S}}{\partial \bar{T}} \right)_{\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4} = \frac{c_1 c_2}{4\Delta A}, \tag{2.26}$$

where c_1, c_2, Δ are given by

$$\begin{aligned}
c_1 &= 2\bar{r}^4 - \left(\sum_{i=1,4} \bar{q}_i^2 \right) \bar{r}^3 + (\bar{q}_1^2\bar{q}_2^2\bar{q}_3^2 + \bar{q}_2^2\bar{q}_3^2\bar{q}_4^2 + \bar{q}_3^2\bar{q}_4^2\bar{q}_1^2 + \bar{q}_4^2\bar{q}_1^2\bar{q}_2^2)\bar{r} - 2 \prod_{i=1,4} \bar{q}_i^2, \\
c_2 &= 3\bar{r}^4 + 2\left(\sum_{i=1,4} \bar{q}_i^2 \right) \bar{r}^3 + (\bar{q}_1^2\bar{q}_2^2 + \bar{q}_1^2\bar{q}_3^2 + \bar{q}_2^2\bar{q}_3^2 + \bar{q}_1^2\bar{q}_4^2 + \bar{q}_2^2\bar{q}_4^2 + \bar{q}_3^2\bar{q}_4^2)\bar{r}^2 - \prod_{i=1,4} \bar{q}_i^2, \\
\Delta &= 3\bar{r}^4 - 2\left(\sum_{i=1,4} \bar{q}_i^2 \right) \bar{r}^3 + (\bar{q}_1^2\bar{q}_2^2 + \bar{q}_2^2\bar{q}_3^2 + \bar{q}_3^2\bar{q}_4^2 + \bar{q}_1^2\bar{q}_3^2 + \bar{q}_1^2\bar{q}_4^2 + \bar{q}_2^2\bar{q}_4^2)\bar{r}^2 - \prod_{i=1,4} \bar{q}_i^2.
\end{aligned} \tag{2.27}$$

The suceptibility in this case is a four dimentional metrix defined as

$$\bar{\chi}_{ij} = \left(\frac{\partial \bar{\rho}_i}{\partial \bar{\mu}_j} \right)_{\bar{T}, \bar{\mu}_k, \bar{\mu}_l, \bar{\mu}_m}, \text{ with } \; k, l, n \neq j. \tag{2.28}$$

Two of the components are

$$\begin{aligned}
\bar{\chi}_{11} &= \left(\frac{\partial \bar{\rho}_1}{\partial \mu_1} \right)_{\bar{T}, \bar{\mu}_2, \bar{\mu}_3, \bar{\mu}_4} = \frac{\bar{r}\delta_1}{32\pi\Delta}, \\
\bar{\chi}_{12} &= \left(\frac{\partial \bar{\rho}_1}{\partial \mu_2} \right)_{\bar{T}, \bar{\mu}_1, \bar{\mu}_3, \bar{\mu}_4} = -\frac{\bar{q}_1\bar{q}_2\delta_2}{32\pi\Delta},
\end{aligned} \tag{2.29}$$

where

$$\delta_1 = 3\bar{r}^4 + 3(2\bar{q}_1^2 - \bar{q}_2^2 - \bar{q}_3^2 - \bar{q}_4^2)\bar{r}^3 + (\bar{q}_2^2\bar{q}_3^2 + \bar{q}_3^2\bar{q}_4^2 + \bar{q}_2^2\bar{q}_4^2 - 3(\bar{q}_1^2\bar{q}_2^2 - \bar{q}_1^2\bar{q}_3^2 - \bar{q}_1^2\bar{q}_4^2)) + 3\prod_{j=1,4}\bar{q}_j^2,$$

and

$$\delta_2 = \bar{r}^4 + 2(\bar{q}_1^2 + \bar{q}_2^2 - \bar{q}_3^2 - \bar{q}_4^2)\bar{r}^3 + (3\bar{q}_3^2\bar{q}_4^2 - \bar{q}_1^2\bar{q}_2^2 - \bar{q}_1^2\bar{q}_3^2 - \bar{q}_2^2\bar{q}_3^2 - \bar{q}_1^2\bar{q}_4^2 - \bar{q}_2^2\bar{q}_4^2)\bar{r}^2 + \prod_{j=1,4}\bar{q}_j^2.$$

Other components of suceptibility can be obtained by cyclically permuting the indices (1234). We note that specific heat and suceptibility diverge at $\Delta = 0$.

Apart from some isolated cases (discussed below), all black holes suffer from instabilities. When some of the charges are either equal or zero, the critical surfaces change as can be seen directly by sustituating charges in the general expressions above. We list down various such critical surfaces below when some of the charges are identically zero:

$$\begin{aligned} 3\bar{r} - 2\bar{q}_1^2 &= 0, && \text{for } \bar{q}_1 \neq 0, \bar{q}_2 = \bar{q}_3 = \bar{q}_4 = 0, \\ 3\bar{r}^2 - 2\bar{r}(\bar{q}_1^2 + \bar{q}_2^2) + \bar{q}_1^2\bar{q}_2^2 &= 0, && \text{for } \bar{q}_1, \bar{q}_2 \neq 0, \bar{q}_3 = \bar{q}_4 = 0. \end{aligned} \tag{2.30}$$

Furthermore, unlike the black holes in five dimensions, these black holes show instability even when two of their charges are equal. In particuler, $\bar{q}_1 = \bar{q}_2 \neq 0$ and with $\bar{q}_3 = \bar{q}_4 = 0$, the singularities appear at

$$3\bar{r} - 2\bar{q}_1^2 = 0. \tag{2.31}$$

For black holes with three charges, the critical surfaces are

$$\begin{aligned} 3\bar{r}^2 - 2\bar{r}(\bar{q}_1^2 + \bar{q}_2^2 + \bar{q}_3^2) + \bar{q}_1^2\bar{q}_2^2 + \bar{q}_1^2\bar{q}_3^2 + \bar{q}_2^2\bar{q}_3^2 &= 0 \quad \text{for } \bar{q}_1, \bar{q}_2, \bar{q}_3 \neq 0, \bar{q}_4 = 0, \\ 3\bar{r} - \bar{q}_1^2 - 2\bar{q}_3^2 &= 0 \qquad \text{for } \bar{q}_1 = \bar{q}_2 \neq 0, \bar{q}_3 \neq 0, \bar{q}_4 = 0. \end{aligned} \tag{2.32}$$

We further note that, for three non-zero charges, black holes are thermodynamically stable only when $\bar{q}_1 = \bar{q}_2 = \bar{q}_3$. Though for generic values of four charges, instabilty appears at $\Delta = 0$, when some of the charges are equal location of thermodynamic singularities change. This is given in the following list.

$$\begin{aligned} 3\bar{r}^3 - \bar{r}^2(2\bar{q}_1^2 + 2\bar{q}_2^2 + \bar{q}_3^2) + \bar{r}\bar{q}_1^2\bar{q}_2^2 + \bar{q}_1^2\bar{q}_2^2\bar{q}_3^2 &= 0 && \text{for } \bar{q}_1 \neq \bar{q}_2, \bar{q}_3 = \bar{q}_4, \\ 3\bar{r}^3 - \bar{r}(\bar{q}_1^2 + \bar{q}_3^2) - q_1^2 q_3^2 &= 0 && \text{for } \bar{q}_1 = \bar{q}_2, \bar{q}_3 = \bar{q}_4, \\ 3\bar{r}^2 - 2\bar{r}q_4^2 - q_1^2 q_4^2 &= 0 && \text{for } \bar{q}_1 = \bar{q}_2 = \bar{q}_3, \bar{q}_4 \neq 0. \end{aligned} \tag{2.33}$$

However, when all the charges are equal, thermodynamic instabilities are *absent*.

Approaching the critical surfaces in different ways, it is possible to find all the critical exponents as in the previous subsection. In all the cases it turns out that the exponents are same as in (2.19).

In the rest of this subsection, we consider the thermodynamic properties of black holes in seven dimensions. Since those black holes correspond to S^4 reduction of spinning $M5$ branes in eleven dimensional supergravity, the Cartan consists of $\mathrm{U}(1) \times \mathrm{U}(1)$. Therefore, a general R-charged black hole can carry only two independent charges. The general form of the solutions are given in [12]. We only list down the thermodynamic variables. The horizon area is proportional to

$$A = \sqrt{(\bar{r}^4 + \bar{q}_1^2)(\bar{r}^4 + \bar{q}_2^2)}, \tag{2.34}$$

The temperature is

$$\bar{T} = \frac{3\bar{r}^8 + \bar{r}^4(\bar{q}_1^2 + \bar{q}_2^2) - \bar{q}_1^2\bar{q}_2^2}{2\pi\bar{r}^3 A} \,. \tag{2.35}$$

Entropy, energy and pressure are

$$\bar{S} = \frac{A}{4r}\,, \qquad \bar{E} = \frac{5A^2}{16\pi}\,, \qquad \bar{P} = \frac{A}{16\pi}. \tag{2.36}$$

The components of the charge and chemical potential are

$$\bar{\rho}_i = \frac{\sqrt{2}\bar{q}_i}{8\pi\bar{r}}A\,, \qquad \bar{\mu}_i = \frac{\bar{q}_i}{\bar{r}(\bar{r}^4 + \bar{q}_i^2)}A. \tag{2.37}$$

The specific heat is

$$\bar{C} = \frac{\bar{r}(5\bar{r}^8 - \bar{r}^4\bar{q}_1^2 - \bar{r}^4\bar{q}_2^2 - 3\bar{q}_1^2\bar{q}_2^2)(3\bar{r}^8 + \bar{r}^4\bar{q}_1^2 + \bar{r}^4\bar{q}_2^2 - \bar{q}_1^2\bar{q}_2^2)}{3\bar{r}^8 - \bar{r}^4\bar{q}_1^2 - \bar{r}^4\bar{q}_2^2 - q_1^2\bar{q}_2^2}. \tag{2.38}$$

The suceptibility is a two dimensional symmetric matrix with entries

$$\begin{aligned}\bar{\chi}_{11} &= \frac{\bar{r}(3\bar{r}^{12} + 13\bar{r}^8\bar{q}_1^2 - \bar{r}^8\bar{q}_2^2 - 4\bar{r}^4\bar{q}_1^4 + \bar{r}^4\bar{q}_1^2\bar{q}_2^2 - 4\bar{q}_1^4\bar{q}_2^4)}{8\pi(3\bar{r}^8 - \bar{r}^4\bar{q}_1^2 - \bar{r}^4\bar{q}_2^2 - \bar{q}_1^2\bar{q}_2^2)}, \\ \bar{\chi}_{12} &= -\frac{\bar{q}_1\bar{q}_2\bar{r}(5\bar{q}_1\bar{q}_2 + 3\bar{r}^4\bar{q}_1^2 + 5\bar{r}^4q_2^2 - 5\bar{r}^8)}{3\bar{r}^8 - \bar{r}^4\bar{q}_1^2 - \bar{r}^4\bar{q}_2^2 - \bar{q}_1^2\bar{q}_2^2}.\end{aligned} \tag{2.39}$$

$\bar{\chi}_{22}$ can be obtained by interchanging the indices of the expression of $\bar{\chi}_{11}$. Thermodynamic instabilities appear for all the black holes unless $\bar{q}_1 = \bar{q}_2$. The critical surfaces are listed below.

$$\begin{aligned}3\bar{r}^8 - \bar{r}^4\bar{q}_1^2 - \bar{r}^4\bar{q}_2^2 - q_1^2\bar{q}_2^2 = 0 \qquad & \text{for } \bar{q}_1 \neq \bar{q}_2 \neq 0, \\ 3\bar{r}^4 - \bar{q}_1^2 = 0, \qquad & \text{for } \bar{q}_1 \neq 0, \bar{q}_2 = 0.\end{aligned} \tag{2.40}$$

Critical exponents turn out to be same as (2.19).

3 Mean field analysis

In the previous section, we studied the thermodynamic properties of R-charged black holes in five, four and seven dimensions. We have seen that, for generic values of charges, these black holes undergo continuous phase transition characterized by certain values of critical exponents. In this section, our aim is to provide a mean field description of this black hole phase transition. More specifically, within the grand canonical ensemble, by identifying the

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horizon radius as an order parameter, we provide Bragg-Williams like effective potentials[1] that are expected to decribe instabilities associated with these black holes.[2] On shell, this potential reduces to the grand canonical potential. Furthermore, we will see, near the critical surfaces, the effective potentials leads to the right set of critical exponents.

This section has three subsections. In the first subsection, we analyze black holes in five dimensions. In particular, we provide the effective potentials for black holes with one or two charges. For the case of three non-zero charges, owing to the complexity of the thermodynamic variables, we have not been able to construct mean field potential in a simple closed form. In the next subsection, we carry out a similar exercise for black holes in four and seven dimensions. Finally, in the last part of this section, by trading horizon radius with charge density of the boundary gauge theory, we propose an effective potential which is expected to decribe the gauge theory in the presence of non-zero chemical potential. We illustrate this explicitly for simplest R-charged black hole, namely, for a singly charged black hole in five dimensions.

3.1 Black holes in five dimensions

We start with the simplest of R-charged black holes - one with a single non-zero charge in five dimensions. Thermodynamic quantities can be found out simply setting $\bar{q}_2 = \bar{q}_3 = 0$ in (2.8), (2.9), (2.10). We suggest that it is described, by an effective potential[3]

$$\bar{\mathcal{F}}(\bar{T}, \bar{\mu}_1, \bar{r}) = \frac{1}{8\pi}\left[\frac{\bar{r}^4(3\bar{r}^2 - \bar{\mu}_1^2)}{2\bar{r}^2 - \bar{\mu}_1^2} - \frac{2\sqrt{2}\pi\bar{r}^4\bar{T}}{\sqrt{2\bar{r}^2 - \bar{\mu}_1^2}}\right]. \tag{3.1}$$

Note that, as in grand canonical ensemble, $\bar{T}, \bar{\mu}_1$ are to be treated as external parameters. $\bar{\mathcal{F}}$ also depends upon $\bar{r}$, the dimensionless horizon radius. $\bar{r}$ will be treated here as the order parameter. As we will discuss below, our choice leads to the correct set of critical exponents near the critical line. We also point out that in $\bar{\mathcal{F}}$, $\bar{r}$ can take any value; the relation between $\bar{r}$ and the mass and the charge of the black hole appears as a consequence of on-shell condition.

The equilibrium condition of the system is represented by

$$\frac{\partial \bar{\mathcal{F}}}{\partial \bar{r}} = 0. \tag{3.2}$$

This leads to

$$\bar{T} = \frac{(4\bar{r}^2 - \bar{\mu}_1^2)}{2\sqrt{2}\pi\sqrt{(2\bar{r}^2 - \bar{\mu}_1^2)}} = \frac{\bar{q}_1 + 2\bar{r}^2}{2\pi\sqrt{\bar{r}^2 + \bar{q}_1}}. \tag{3.3}$$

---

[1]Similar constructions were also found useful in oreder to study various black holes and gauge theories around the Hawking-Page points. See for example [16–18].

[2]Though in this paper, we will interchangeably use the words Bragg-Williams potential and mean-field potential, it should be recognized that BW potential and Landau mean-field potential are not the same. Close to T_c, the later appears, in some cases, as an expansion of the former in terms of order paramer. We will comment on such a phenomenon in section 3.3.

[3]Constructional procedure is similar to the one described in [10].

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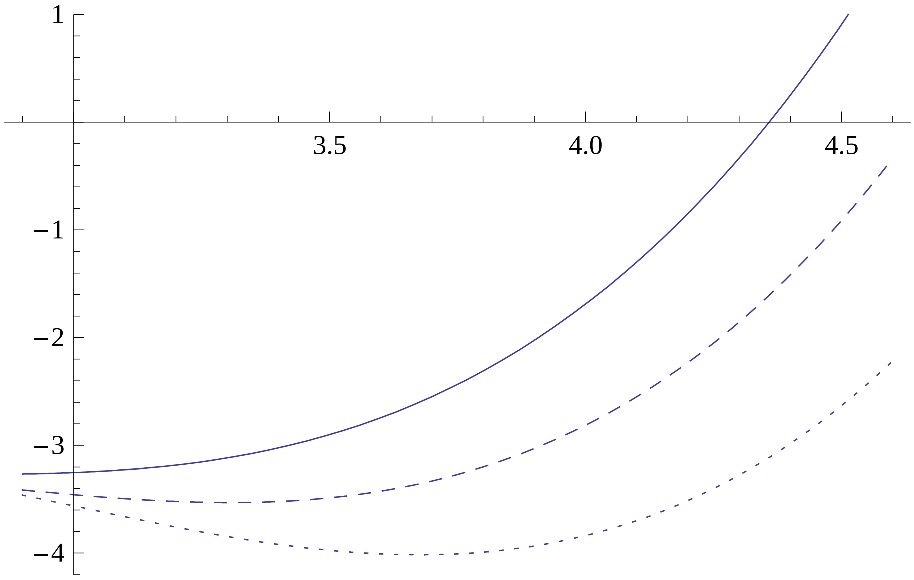

Figure 1. $\bar{\mathcal{F}}$ as a function of order parameter $\bar{r}$ for fixed $\bar{\mu}_1$. The solid line is for the $\bar{T} = \bar{T}_c$. The other two lines are for $\bar{T} > \bar{T}_c$. As $\bar{T}$ increases, minimum of $\bar{\mathcal{F}}$ appears for larger values of $\bar{r}$. From $\bar{T} = \bar{T}_c$, the order parameter changes continuously with $\bar{T}$.

This is the temperature of a singly charged black hole as can be seen from (2.8) after substitutions $\bar{q}_2 = \bar{q}_3 = 0$. Now using (3.3) in (3.1), we get

$$\bar{\mathcal{F}} = \frac{\bar{r}^6}{8\pi(\bar{\mu}_1^2 - 2\bar{r}^2)} = -\frac{\bar{r}^2(\bar{q}_1 + \bar{r}^2)}{16\pi} = -\bar{P}. \tag{3.4}$$

So on-shell, $\bar{\mathcal{F}}$ reduces to the negative of the pressure $\bar{P}$ which is same as the grand canonical potential. The extrema of (3.1) appear at

$$\begin{aligned} \bar{r} &= 0, \\ &= \sqrt{\frac{2}{3}}\bar{\mu}_1, \\ &= \frac{1}{2}\sqrt{\bar{\mu}_1^2 + 2\pi^2\bar{T}^2 - 2\sqrt{\pi^4\bar{T}^4 - \pi^2\bar{\mu}_1^2\bar{T}^2}}, \\ &= \frac{1}{2}\sqrt{\bar{\mu}_1^2 + 2\pi^2\bar{T}^2 + 2\sqrt{\pi^4\bar{T}^4 - \pi^2\bar{\mu}_1^2\bar{T}^2}}. \end{aligned} \tag{3.5}$$

We note that within the range of our interest, namely for $\bar{q}_1 \leq 2\bar{r}^2$ (which leads to $\bar{T} \geq \bar{T}_c$) , fourth root of (3.5) corresponds to the only minimum. This equilibrium value of $\bar{r}$ increases with the temperature.

Behaviour of $\bar{\mathcal{F}}$ as a function of the order parameter for fixed $\bar{\mu}_1$ and for different temperatures is shown in the figure 1. Clearly, the order parametr $\bar{r}$ changes continuously around the critical temperature. This is typical of a second order phase transition.

The mean field potential $\bar{\mathcal{F}}$ reporduces correct set of critical exponents discussed in section 2.1. Approaching the critical line along constant $\bar{\mu}_1 = \bar{\mu}_{1c}$, we see from the last

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root of (3.5)

$$\bar{r} - \bar{r}_c \sim (\bar{T} - \bar{T}_c)^{\frac{1}{2}}, \text{ with } \bar{r}_c = \frac{\sqrt{3}}{2}\bar{\mu}_{1c}. \tag{3.6}$$

This leads to the critical exponent $\beta = \frac{1}{2}$. Similarly, the susceptibility, that follows from (3.5), behaves as

$$\bar{\chi} = \frac{\partial \bar{r}}{\partial \bar{\mu}_1}|_{\bar{T}} \sim (\bar{T} - \bar{T}_c)^{-\frac{1}{2}}, \tag{3.7}$$

near the critical temperature. Therefore $\gamma = \frac{1}{2}$. The specific heat follows from (3.1).

$$\bar{C}_{\bar{\mu}_1} = -\bar{T}\frac{\partial^2 \bar{\mathcal{F}}}{\partial T^2}|_{\bar{\mu}_1} \sim (\bar{T} - \bar{T}_c)^{-\frac{1}{2}}, \tag{3.8}$$

which gives $\alpha = \frac{1}{2}$. Finally, approaching the critical line with $\bar{T} = \bar{T}_c$ and using (2.10), we get from the last root of (3.5)

$$\bar{r} - \bar{r}_c \sim -\ (\bar{\mu}_{1c} - \bar{\mu}_1)^{\frac{1}{2}}, \text{ with } \bar{r}_c = \frac{\pi\sqrt{3}}{2}\bar{T}_c. \tag{3.9}$$

where $\bar{r}_c$ is the critical value of $\bar{r}$ at $\bar{\mu}_1 = \bar{\mu}_{1c}$. Hence, we have $\delta = 2$. We therefore conclude that the effective potential of (3.1) reproduces all the critical exponents given in (2.19).

Next, we consider black holes with two non-zero charges. Thermodynamic variables can be found from (2.8), (2.9), (2.10) by substituting $\bar{q}_3 = 0$. The mean field potential can be expressed in the following manner:

$$\bar{\mathcal{F}}(\bar{T}, \bar{\mu}_1, \bar{\mu}_2, \bar{r}) = \frac{(\bar{\mu}_1^2\bar{\mu}_2^2 - 4\bar{r}^4)}{8\pi} \times \left[\frac{(12\bar{r}^4 - 4\bar{\mu}_1^2\bar{r}^2 - 4\bar{\mu}_2^2\bar{r}^2 + \bar{\mu}_1^2\bar{\mu}_2^2)}{8(\bar{\mu}_1^2 - 2\bar{r}^2)(\bar{\mu}_2^2 - 2\bar{r}^2)} - \frac{\bar{T}\bar{r}\pi}{\sqrt{(\bar{\mu}_1^2 - 2\bar{r}^2)(\bar{\mu}_2^2 - 2\bar{r}^2)}}\right]. \tag{3.10}$$

The potential is a function of the order parameter $\bar{r}$ and also depends on the external parameters $\bar{T}, \bar{\mu}_1$ and $\bar{\mu}_2$. Note that in the limit $\bar{\mu}_2 = 0$, $\bar{\mathcal{F}}$ reduces to that of (3.1).

The equilibrium configuration is found by minimizing the potential with respect to $\bar{r}$. This leads to

$$\bar{T} = \frac{\bar{r}(4\bar{r}^2 - \bar{\mu}_1^2 - \bar{\mu}_2^2)}{2\pi\sqrt{(\bar{\mu}_1^2 - 2\bar{r}^2)(\bar{\mu}_2^2 - 2\bar{r}^2)}} = \frac{\bar{r}}{2\pi}\frac{\bar{q}_1 + \bar{q}_2 + 2\bar{r}^2}{\sqrt{(\bar{r}^2 + \bar{q}_1)(\bar{r}^2 + \bar{q}_2)}}. \tag{3.11}$$

This is the equilibrium temperature of the doubly charged black hole (2.8). Substituting (3.11) in (3.10), we get

$$\bar{\mathcal{F}} = -\frac{(\bar{\mu}_1^2\bar{\mu}_2^2 - 4\bar{r}^4)^2}{64\pi(\bar{\mu}_1^2 - 2\bar{r}^2)(\bar{\mu}_2^2 - 2\bar{r}^2)} = -\frac{1}{16\pi}(\bar{r}^2 + \bar{q}_1)(\bar{r}^2 + \bar{q}_2) = -\bar{P}. \tag{3.12}$$

The values of $\bar{r}$ at which $\bar{\mathcal{F}}$ has extrema can be evaluated analytically. However, expressions are very large and unilluminating. We do not display them here. It turns out that in the region of our interest namely for $2\bar{r}^2 - \bar{q}_1 - \bar{q}_2 \geq 0$, there is only a single minimum of $\bar{\mathcal{F}}$. Typical behaviour of $\bar{\mathcal{F}}$ as a function of $\bar{r}$ for fixed $\bar{\mu}_1$ and $\bar{\mu}_2$ is shown in the figure 2.

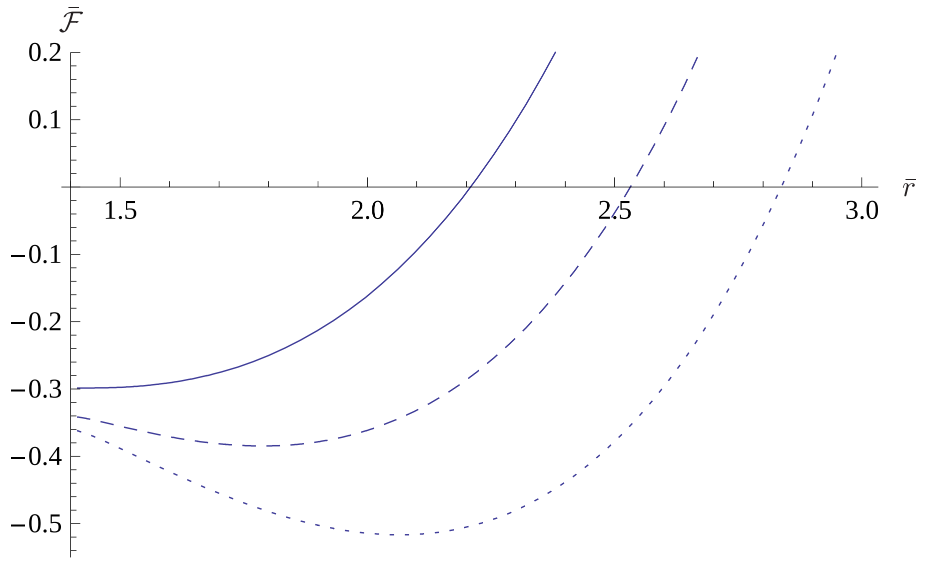

Figure 2. $\bar{\mathcal{F}}$ as a function of order parameter $\bar{r}$ for fixed $\bar{\mu}_1$ and $\bar{\mu}_2$. The solid line is for the $\bar{T} = \bar{T}_c$. The other two lines are for $\bar{T} > \bar{T}_c$. As $\bar{T}$ increases, minimum of $\bar{\mathcal{F}}$ appears for larger values of $\bar{r}$. From $\bar{T} = \bar{T}_c$, the order parameter changes continuously with $\bar{T}$

We see that the order parameter $\bar{r}$, at which the minimum appears, continuously increases as we increase the temperature. This shows that the phase transition is of second order.

Various critical exponents can be evaluated as in the single charged case. Approaching the critical surface along constant $\bar{\mu}_1$ and $\bar{\mu}_2$, and expanding the minimum value of $\bar{r}$ close to the critical temperature, we get

$$\bar{r} - \bar{r}_c \sim (\bar{T} - \bar{T}_c)^{\frac{1}{2}}. \tag{3.13}$$

Similarly, on constant $\bar{T}, \bar{\mu}_2$ and $\bar{T}, \bar{\mu}_1$ surfaces, we get

$$\bar{\chi}_1 = \frac{\partial \bar{r}}{\partial \bar{\mu}_1} \sim (\bar{T} - \bar{T}_c)^{-\frac{1}{2}}, \qquad \bar{\chi}_2 = \frac{\partial \bar{r}}{\partial \bar{\mu}_2} \sim (\bar{T} - \bar{T}_c)^{-\frac{1}{2}}, \tag{3.14}$$

respectively. The specific heat follows from taking second derivative of $\bar{\mathcal{F}}$ with respect to $\bar{T}$ for fixed $\bar{\mu}_1$ and $\bar{\mu}_2$. This leads to

$$C_{\bar{\mu}_1, \bar{\mu}_2} = -\bar{T} \frac{\partial^2 \bar{\mathcal{F}}}{\partial \bar{T}^2} \sim (\bar{T} - \bar{T}_c)^{-\frac{1}{2}}. \tag{3.15}$$

Finally, on constant $\bar{T}, \bar{\mu}_2$ and $\bar{T}, \bar{\mu}_1$ surfaces, expanding around the minimum of $\bar{\mathcal{F}}$ for $\bar{\mu}_1$ and $\bar{\mu}_2$ close to their critical values, we get

$$\bar{r} - \bar{r}_c \sim -\, (\bar{\mu}_1 - \bar{\mu}_{1c})^{\frac{1}{2}}, \qquad \bar{r} - \bar{r}_c \sim -\, (\bar{\mu}_2 - \bar{\mu}_{2c})^{\frac{1}{2}}, \tag{3.16}$$

respectively. We now note that these lead to the same critical exponents that we got from our analysis in the previous section, see (2.19).

3.2 Black holes in four and seven dimensions

Here we reconsider the black holes in four and seven dimensions. As in previous subsection, our aim will be to understand their behaviours near the critical surfaces in terms of effective potentials. Though for all non-zero charges, it is in principle possible to construct the function $\bar{\mathcal{F}}$, it becomes computationally difficult to express $\bar{\mathcal{F}}$ in terms of temperature and all non-zero chemical potentials in a closed form. We will therefore consider, in this subsection, black holes with a single charge.

Thermodynamic variables for four dimensional holes with single charge can be found out by substituting $\bar{q}_2 = \bar{q}_3 = \bar{q}_4 = 0$ in (2.25). The effective potential can be written down as

$$\bar{\mathcal{F}}(\bar{T}, \bar{\mu}_1) = \frac{1}{16\pi}\left[\frac{\bar{r}^3(4\bar{r}^2 - \bar{\mu}_1^2)}{2\bar{r}^2 - \bar{\mu}_1^2} - 4\sqrt{2}\pi\bar{T}\bar{r}^3\sqrt{\frac{1}{2\bar{r}^2 - \bar{\mu}_1^2}}\right]. \tag{3.17}$$

The saddle point of $\bar{\mathcal{F}}$ gives the on-shell temperature of the black hole

$$\bar{T} = \frac{6\bar{r}^2 - \bar{\mu}_1^2}{4\sqrt{2}\pi\sqrt{2\bar{r}^2 - \bar{\mu}_1^2}} = \frac{(3\bar{r} + 2\bar{q}_1^2)\sqrt{\bar{r}}}{4\pi\sqrt{\bar{r} + \bar{q}_1^2}}. \tag{3.18}$$

This is same as what we would have gotten from (2.25) after setting three charges to zero.

The critical line (2.31), while expressed in terms of $\bar{T}$ and $\bar{\mu}_1$ reads as $\bar{T}_c = \sqrt{3}\bar{\mu}_1/2\pi$. We will be interested in the rigion $\bar{T} \geq \bar{T}_c$ or equvalently in the region $\bar{q}_1^2/r \leq 3/2$. There is only one minimum of $\bar{\mathcal{F}}$ in this region and is given by

$$\bar{r} = \sqrt{\frac{\bar{\mu}_1^2}{6} + \frac{8\pi^2\bar{T}^2}{9} + \frac{4}{9}\sqrt{4\pi^4\bar{T}^4 - 3\pi^2\bar{\mu}_1^2\bar{T}^2}}. \tag{3.19}$$

Studying $\bar{\mathcal{F}}$ around this point, as in the last subsection, we recover the critical exponents (2.19).

For seven dimensional black holes, one can similarly construct the effective potential $\bar{\mathcal{F}}$. Following is the expression of $\bar{\mathcal{F}}$ for single charge case.

$$\bar{\mathcal{F}}(\bar{r}, \bar{\mu}_1) = \frac{\bar{r}^6}{8\pi}\left[\frac{5\bar{r}^2 - 2\mu_1^2}{2\bar{r}^2 - \mu_1^2} - 2\sqrt{2}\pi\bar{T}\sqrt{\frac{1}{2\bar{r}^2 - \mu_1^2}}\right]. \tag{3.20}$$

As before at the saddle point this function reduces to grand canonical potential with temperature given by the one in (2.37) with $\bar{q}_2 = 0$. The critical surface appears at $\bar{T} = \sqrt{\frac{3}{2}}\frac{\bar{\mu}_1}{\pi}$. For a given $\bar{\mu}_1$, this equation defines $\bar{T}_c$. In the range of our interests, namely for $\bar{T} \geq \bar{T}_c$, there is only one minimum of $\bar{\mathcal{F}}$. This is given by

$$\bar{r} = \sqrt{\frac{\mu_1^2}{3} + \frac{2\pi^2\bar{T}^2}{9} + \frac{1}{9}\sqrt{4\pi^4\bar{T}^4 - 6\pi^2\bar{\mu}_1^2\bar{T}^2}}. \tag{3.21}$$

Studying the behaviour of $\bar{\mathcal{F}}$ around this point, we recover all the exponents that are given in (2.19).

3.3 Proposal for gauge theory effective potentials

We now wish to use the results of the previous subsections to propose an effective potential describing the gauge theory. While it might be possible to construct such a potential in complete generality, in this subsection we study the simplest case. We consider the four dimensioal finite temperature gauge theory where only one chemical potential is turned on.

The right order parameter to work with, at the boundary, is the R-charge density ρ_1. The conjugate chemical potential is μ_1. We can now use the first equation of (2.10) to express (3.1) in terms of ρ_1. This leads to the following equation for $\mathcal{F}$:

$$\begin{aligned}\mathcal{F} = \ & -\mu_1\rho_1 - \frac{6\pi\rho_1}{N_c^2\mu_1^2}\sqrt{4\pi^2\rho_1^2 - N_c^2\mu_1^3\rho_1} + \frac{12\pi^2\rho_1^2}{N_c^2\mu_1^2} \\ & -\frac{2\pi\rho_1}{\mu_1}T\sqrt{-2\mu_1^2 + \frac{16\pi^2\rho_1}{\mu_1 N_c^2} - \frac{8\pi}{\mu_1 N_c^2}\sqrt{4\pi^2\rho_1^2 - N_c^2\mu_1^3\rho_1}}.\end{aligned} \tag{3.22}$$

In writing this down, we have reinstated G the five dimensioal gravitational constant, various factors of l and also used the relation $G = \pi l^3/2N_c^2$. Here N_c is the number of colours.

One can now check that, for $T > T_c = \pi/\mu_{1c}$, the minimum of $\mathcal{F}$ occurs at

$$\rho_1 = \left(\frac{N_c^2\mu_1}{32\pi^2}\right)\frac{\{\mu_1^2 + 2(\pi^2T^2 + \sqrt{\pi^4T^4 - \mu_1^2\pi^2T^2})\}^2}{-\mu_1^2 + 2(\pi^2T^2 + \sqrt{\pi^4T^4 - \mu_1^2\pi^2T^2})}. \tag{3.23}$$

Now expanding (3.22) near $T = T_c$, for fixed μ_1, we get

$$\rho_1 - \rho_{1c} \sim (T - T_c)^{\frac{1}{2}}, \tag{3.24}$$

where the critical charge density is given by $\rho_{1c} = 9\mu_{1c}^3 N_c^2/(32\pi^2)$. The above equation leads to $\beta = 1/2$. Similarly, one can find the other critical exponents given in (2.19).

To this end, we would like to make a couple of comments. Firstly, one may wish to expand $\mathcal{F}$ near $T = T_c$ in powers of $(\rho - \rho_c)/\rho_c$ to get a Ginzburg-Landau potential as an expansion in terms of order parameter. However, in doing so one does not reproduce the right exponents. Higher and higher powers seem to conspire in order to recover the results of (2.19). Secondly, since the gauge theory is a strongly coupled, directly computing the potential (3.22) seems a difficult task. However, in [2], an attempt was made to construct Ginzburg-Landau potential for the system via a regulated field theory model. Unfortunately, all the critical exponents predicted by supergravity configuration could not be recovered.

We end this section with a note that, at least for single charge holes, similar gauge potentials can be easily constructed in three and six dimensions simply using (3.17) and (3.20) along with the thermodynamic relations listed in the previous section.

4 R-charge conductivity

So far we have discussed equilibrium thermodynamics of the black holes and their dual gauge theories. In this section we will turn towards non-equilibrium thermodynamics of this system; in particular we will compute the R-charge diffusion constant using gauge-gravity duality. There are two ways to compute R-charge diffusion constant. One can consider retarded correlation function of R-current and look for a pole in the q^2-plane where q represents the momentum. Or one can make use of Green- Kubo formula to calculate the R-charge conductivity λ and find R-charge diffusion coefficient D from the relation $\lambda = D\ .\ \chi$, where χ is the susceptibility matrix. If the susceptibility matrix is already known, the advantage with the the second method is one need to compute the correlator only at the zero momentum limit and upto first order in frequency. Since, in the present text we have already evaluated the susceptibility matrix for various cases we will consider the latter method only.

The procedure of computing R-charge diffusion constant using Kubo formula is explained in detail in [19, 20]. For convenience of the reader we recount the essential steps involved in the following. For this method we need to find out the retarded correlator of the R-current:

$$\tilde{G}^{(R)}_{xx}(\omega,\vec{q}) = -i\int_{-\infty}^{\infty} dt\ e^{i\omega t}\ \theta(t)\int d\vec{x}e^{-i\vec{q}.\vec{x}}\left\langle[J_x(t,\vec{x}),J_x(0,\vec{0})]\right\rangle . \tag{4.1}$$

at the zero momentum limit. The R-charge conductivity λ is given in terms of retarded correlator by Green-Kubo formula:

$$\begin{aligned}\lambda &= -\lim_{\omega\to 0}\frac{\Im[\tilde{G}^{(R)}_{xx}(\omega,\tilde{q}=0)]}{\omega}\\ &= \lim_{\omega\to 0}\frac{1}{2\omega}\int_{-\infty}^{\infty} dt\ e^{i\omega t}\int d\vec{x}\left\langle[J_x(t,\vec{x}),J_x(0,\vec{0})]\right\rangle .\end{aligned}$$

The retarded correlator can be computed as follows: We find perturbations in various modes around the black hole solution that satisfy linearized equation following from equations of motion and the 'incoming wave boundary condition' at the horizon. Then we calculate the boundary action which can be obtained by plugging in the solutions in the expression of total action evaluated at boundary. The total action consists of three pieces: the bulk action, the Gibbons-Hawking term and the counterterm required to cancel the divergences. The retarded correlator can be obtained by taking derivative of the boundary action with respect to boundary value (of the perturbation in the gauge field) twice.

Let us demonstarte it in the case of five dimensional black hole with three charges. For the present purpose it is sufficient to consider perturbations in the tensor (metric) and the vector (gauge fields) modes around the black hole solution and keep the scalars unperturbed. So perturbations are of the form:

$$g_{\mu\nu} = \mathbf{g}^{(0)}_{\mu\nu} + h_{\mu\nu}\,, \qquad A^i_\mu = \mathbf{A}^{i(0)}_\mu + \mathcal{A}^i_\mu\,, \quad i=1,2 \quad . \tag{4.2}$$

In order to determine R-charge conductivity it is enough to consider perturbations in (tx) and (xx) component of the metric tensor and x component of the gauge fields. Moreover one can choose the perturbations to depend on radial coordinate r, time t and one of the spatial worldvolume coordinate z.

A convenient ansatz with the above restrictions in mind is:[4]

$$h_{tx} = \mathbf{g}_{0xx}\ T(u)\ e^{-i\omega t+iqz}, \quad h_{zx} = \mathbf{g}_{0xx}\ Z(u)\ e^{-i\omega t+iqz}, \quad \mathcal{A}^i_x = \frac{\mu}{2}\ A^i(u)\ \ e^{-i\omega t+iqz}. \tag{4.3}$$

Here ω and q represent the freeqeuncy and momentum in z direction respectively and we set perturbations in the other components to be equal to zero.

Next step is to find linearized equations following from the equations of motion at zero momentum limit that the perturbations has to satisfy. It turns out that in linearized equation at zero momentum limit metric perturbation $Z(u)$ decouple from the rest. One can further eliminate $T(u)$ reducing it to equation for perturbations in gauge field only. For five dimensional black hole with all the three charges the equation becomes in this notation

$$A_1'' + \left(\frac{f'}{f} + \frac{H_1'}{H_1} - \frac{H_2'}{H_2} - \frac{H_3'}{H_3}\right) A_1' + \frac{\tilde{\omega}^2\mathcal{H}}{f^2 u} A^1 - \frac{u(1+k_1)}{fH_1^2}\Big[k_1(1+k_2)(1+k_3)A_1 \\ + k_2(1+k_1)(1+k_3)A_2 + k_3(1+k_1)(1+k_2)A_3\Big] = 0, \tag{4.4}$$

while equation for the other two components can be obtained by cyclically permuting (123).

We require solutions to these equations with “incoming wave boundary condition” at the horizon at $u = 1$. Since $f(u)$ vanishes at $u = 1$ one can solve the indicial equation near $u = 1$ and obtain two solutions which have asymptotic behaviour near $u = 1$ as $(u-1)^\nu$ with $\nu = \pm i\tilde{\omega}(T_0/2T)$. The incoming wave function corresponds to the negative value and so incorporating the boundary condition corresponds to the following ansatz:

$$A_i = \frac{f^{-i\tilde{\omega}(T_0/2T)}}{1+k_i u} a_i(u), \qquad i = 1, 2, 3. \tag{4.5}$$

Since in the sequel we need solutions upto first order of ω only we can write it as

$$a_i(u) = p_i(u) + i\omega s_i(u) + 0(\omega^2). \tag{4.6}$$

In terms of a_i's the boundary action appears to be

$$\mathrm{S}_{\text{boundary}} = \lim_{u\to 0} \frac{N_c^2 T_0^2}{16} \int\ dtd\vec{x}\ \frac{f}{\mathcal{H}}[H_1^2 a_1' a_1 + H_2^2 a_2' a_2 + H_3^2 a_3' a_3], \tag{4.7}$$

where we have kept terms quadratic in perturbation. The retarded correlator can be obtained by taking derivative of the total action with respect to boundary value twice.

Proceeding further with this generic charge leads to expressions, which are technically complicated. Moreover, one unusual problem here is that we cannot obtain special cases,

[4]Here is a comment about notation in this section. To facillitate comparison with expressions in existing literature we have made a few changes in our notation. In what follows, instead of r we use $u = (r_+/r)^2$ so the boundary is at $u = 0$ and horizon is at $u = 1$. Similarly the charge parameters q_i will be traded for $k_i = (q_i/r_+^2)$ and we will also introduce $T_0 = (\frac{r_+}{l^2\pi}$. In this notation the solution for black hole is given by $H_i = (1 + k_i u)$, $\mathcal{H} = \prod_{i=1}^m H_i$ where m is number of charges. For five dimensional black hole $f = \mathcal{H} - \prod_{i=1}^m (1 + k_i)u^2$ and for four or sweven dimensional black hole $f = \mathcal{H} - \prod_{i=1}^m (1 + k_i)u^3$. We will also use $\tilde{\omega} = \omega/(2\pi T_0)$

such as two charge solution, starting from generic cases (three charge) by setting one of the charges to zero. This happens because obtaining these equations involves rescaling of fields $A^i(u)$ with respective charges and forces us to do the analysis in each case separately. Therefore, we will restrict ourselves to the R-charge conductivity (for black holes in five, four and seven dimensions) in the simplest cases only, deferring analysis of more complicated cases to the appendix. We will also adjoin a brief outline of the general method that has been used to solve all these cases.

4.1 Five dimensional black hole with two charges

In this case we have two U(1) gauge fields. The solution is

$$H_1 = (1 + k_1 u), \quad H_2 = (1 + k_2 u), \quad \mathcal{H} = H_1 H_2 \quad f(u) = \mathcal{H} - (1 + k_1)(1 + k_2)u^2. \tag{4.8}$$

The perturbed equations are given by,

$$\begin{aligned} &A_1'' + \left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}} + 2\frac{H_1'}{H_1}\right) A_1' + \frac{\tilde{\omega}^2 \mathcal{H}}{u f^2} A_1 + \\ &-\frac{u(1+k_1)}{f H_1^2}(k_1(1+k_2)A_1 + k_2(1+k_1)A_2) = 0, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &A_2'' + \left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}} + 2\frac{H_2'}{H_2}\right) A_2' + \frac{\tilde{\omega}^2 \mathcal{H}}{u f^2} A_2 + \\ &-\frac{u(1+k_2)}{f H_2^2}(k_1(1+k_2)A_1 + k_2(1+k_1)A_2) = 0. \end{aligned} \tag{4.10}$$

. We observe that the equation for A_2 can be obtained from equation for A_1 by interchanging k_1, H_1, A_1 with k_2, H_2, A_2 respectively. From now on for convenience, we will write perturbed equation for only one of the gauge fields. The ansatz is given by

$$A_i = \frac{f^{-i\tilde{\omega}(T_0/2T)}}{1 + k_i u} a_i(u), \qquad i = 1, 2, 3. \tag{4.11}$$

We need to solve for $a_i(u)$ in terms of the boundary value of the gauge fields. Since in the sequel we will need solutions upto first order of ω only we can write it as

$$a_i(u) = \left[p_i(u) + i\tilde{\omega} s_i(u) + 0(\tilde{\omega}^2)\right]. \tag{4.12}$$

where i takes value upto 2 for this particular case. It turns out from (4.9), that p_1 and s_1 satisfy the following equations respectively:

$$\begin{aligned} &p_1''(u) + \left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}}\right) p_1'(u) + \\ &-\frac{k_1}{H_1}\left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}}\right) p_1(u) - u\frac{1+k_1}{f H_1}\left[\frac{k_1(1+k_2)}{H_1} p_1 + \frac{k_2(1+k_1)}{H_2} p_2\right] = 0, \\ &s_1''(u) + \left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}}\right) s_1'(u) - \frac{T_0}{T}\frac{f'}{f} p_1'(u) + \frac{T_0}{2T}\left(\frac{f'\mathcal{H}'}{f\mathcal{H}} - \frac{f''}{f}\right) p_1 + \\ &-\frac{k_1}{H_1}\left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}}\right) s_1(u) - u\frac{1+k_1}{f H_1}\left[\frac{k_1(1+k_2)}{H_1} s_1 + \frac{k_2(1+k_1)}{H_2} s_2\right] = 0. \end{aligned}$$

Similarly by interchanging k_1, p_1, s_1 with k_2, p_2, s_2 we obtain equation for p_2 and s_2. According to our boundary condition all these solutions need to be regular at $u = 1$. Without any loss of generality we can choose $s_i = 0$.

The solution for $p_i(u)$ are given below:

$$
\begin{aligned}
p_1(u) &= b_{10}\left(1 + \frac{k_1 u}{2}\right) - b_{20}\frac{k_2}{2}\frac{1+k_1}{1+k_2}u, \\
p_2(u) &= b_{20}\left(1 + k_2 u2\right) - b_{10}\frac{k1}{2}\frac{1+k2}{1+k1}u,
\end{aligned} \tag{4.13}
$$

where b_{10} and b_{20} are boundary value of A_1 and A_2 respectively. The solutions of $s_i(u)$ are more involved and given by

$$
\begin{aligned}
s_1(u) &= h_1 u + (h_2 + h_3 u)\log\left[1 + (1 + k_1 + k_2)u\right], \\
s_2(u) &= l_1 u + (l_2 + l_3 u)\log\left[1 + (1 + k_1 + k_2)u\right].
\end{aligned} \tag{4.14}
$$

Note that at $u = 0$, $s_i(u) = 0$ as mentioned before. The constants are given by

$$
h_1 = b_{10}J_{10} + b_{20}J_{20}, \qquad h_2 = b_{10}L_{10} + b_{20}L_{20}, \tag{4.15}
$$

where

$$
\begin{aligned}
J_{10} &= k_1\left[\frac{(k_1^3 - k_1^2(k_2-3) + k_2(k_2^2 - k_2 - 6) + k_2(k_2^2 + 3k_2 + 2)}{4(1+k_1)(1+k_2)(1+k_1+k_2)}\right], \\
J_{20} &= -k_2\left[\frac{k_1^3 - k_1^2(k_2-3) - k_1(k_2^2 + 2k_2 - 2) + k_2(k_2^2 + 3k_2 + 2)}{4(1+k_2)^2(1+k_1+k_2)}\right].
\end{aligned}
$$

The values of L_{10} and L_{20} are

$$
\begin{aligned}
L_{10} &= \left[\frac{2 + 2k_1^2 + 2k_2 + k_2^2 + k_1(4 + 3k_2)}{(1+k_1+k_2)^2}\right], \\
L_{20} &= -\left[\frac{(1+k_1)k_2(2+k_1+k_2)}{(1+k_2)(1+k_1+k_2)^2}\right].
\end{aligned} \tag{4.16}
$$

h_3 is given by

$$
h_3 = b_{10}k_1 - \frac{k_2(1+k_1)}{1+k_2}b_{20}. \tag{4.17}
$$

l_1, l_2 and l_3 are given by

$$
l_1 = b_{10}\tilde{J}_{10} + b_{20}\tilde{J}_{20}, \qquad l_2 = b_{10}\tilde{L}_{10} + b_{20}\tilde{L}_{20}, \tag{4.18}
$$

where $\tilde{J}_{10}(\tilde{J}_{20})$ can be obtained by interchanging k_1 with k_2 in $J_{20}(J_{10})$ and $\tilde{L}_{10}(\tilde{L}_{20})$ can be obtained by interchanging k_1 with k_2 in $L_{20}(L_{10})$

The second step consists of evaluation of the boundary action quadratic in the boundary values. The boundary action turns out to be of the following form. Since we need only

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imaginary contribution of $0(\tilde{\omega})$ we keep the other terms implicit.

$$\begin{aligned}
\mathrm{S}_{\text{boundary}} &= \lim_{u\to 0}\frac{N_c^2T_0^2}{16}\int dtd\vec{x}\,\frac{f}{\mathcal{H}}\left[H_1^2a_1'a_1+H_2^2a_2'a_2\right]\\
&=\mathcal{O}(\tilde{\omega}^0)+\frac{N_c^2T_0^2}{16}(-i\tilde{\omega})\Big[b_{10}^2\left[J_{10}+\tilde{J}_{10}(1+k_1+k_2)-(T_0/2T)(k_1+k_2)\right]\\
&\quad+b_{10}b_{20}\left[J_{20}\tilde{L}_{20}+(1+k_1+k_2)(\tilde{J}_{20}+\tilde{L}_{20})\right]\\
&\quad+b_{20}^2\left[L_{10}+\tilde{L}_{10}(1+k_1+k_2)-(T_0/2T)(k_1+k_2)\right]\Big]+\mathcal{O}(\tilde{\omega}^2)
\end{aligned}$$

From the above boundary action we obtain the following components of the retarded Green's function

$$\begin{aligned}
\tilde{G}_{11}^{(RR)}(\tilde{\omega},\vec{q}=0) &=\mathcal{O}(\tilde{\omega}^0)-i\tilde{\omega}\frac{N_c^2T_0^2}{8}\left[J_{10}+\tilde{J}_{10}(1+k_1+k_2)-(T_0/2T)(k_1+k_2)\right]+\mathcal{O}(\tilde{\omega}^2),\\
\tilde{G}_{12}^{(RR)}(\tilde{\omega},\vec{q}=0) &=\mathcal{O}(\tilde{\omega}^0)-i\tilde{\omega}\frac{N_c^2T_0^2}{16}\left[J_{20}\tilde{L}_{20}+(1+k_1+k_2)(\tilde{J}_{20}+\tilde{L}_{20})\right]+\mathcal{O}(\tilde{\omega}^2),\\
\tilde{G}_{22}^{(RR)}(\tilde{\omega},\vec{q}=0) &=\mathcal{O}(\tilde{\omega}^0)-i\tilde{\omega}\frac{N_c^2T_0^2}{8}\left[L_{10}+\tilde{L}_{10}(1+k_1+k_2)-(T_0/2T)(k_1+k_2)\right]+\mathcal{O}(\tilde{\omega}^2).
\end{aligned} \tag{4.19}$$

The components of the R-charge conductivity then obtained as

$$\begin{aligned}
\lambda_{11} &=2.\frac{N_c^2T_0^2}{16}\left[J_{10}+\tilde{J}_{10}(1+k_1+k_2)-(T_0/2T)(k_1+k_2)\right]\\
\lambda_{12} &=\frac{N_c^2T_0^2}{16}\left[J_{20}\tilde{L}_{20}+(1+k_1+k_2)(\tilde{J}_{20}+\tilde{L}_{20})\right]\\
\lambda_{22} &=2.\frac{N_c^2T_0^2}{16}\left[L_{10}+\tilde{L}_{10}(1+k_1+k_2)-(T_0/2T)(k_1+k_2)\right].
\end{aligned} \tag{4.20}$$

The susceptibility matrix χ_{ij} can be written in this notation as

$$\chi_{ij}=\frac{\bar{r}^2}{16\pi}\begin{pmatrix} k_1^2+k_1k_2+k_2-5k_1-2 & \sqrt{k_1k_2}(k_1+k_2)\\ \sqrt{k_1k_2}(k_1+k_2) & k_2^2+k_2k_1+k_1-5k_2-2\end{pmatrix}\frac{1}{k_1+k_2-2} \tag{4.21}$$

The components of the diffusion coefficient can be obtained by using the relation $\mathcal{D}=\lambda.\chi^{-1}$. Since all the components of susceptibility diverges on the critical line $k_1+k_2=2$ and conductivity remains finite it follows that all the components of the diffusion coeficient vanish. Vanishing of the diffusion coefficients is consistent with slowing down at critical point.

4.2 Four dimensional black holes

Next we consider the black holes in three dimensions with two charges (single charged case has already been discussed in [7]). There can be maximum four charges as there are four commuting U(1)s. Leaving the more general cases for the appendix, here we consider black holes with two U(1)s. The equation for gauge field is given by

$$\begin{aligned}
&A_1''+\left(\frac{f'}{f}-\frac{\mathcal{H}'}{\mathcal{H}}+2\frac{H_1'}{H_1}\right)A_1'+\frac{\tilde{\omega}^2\mathcal{H}}{uf^2}A_1+\\
&-\frac{u^2(1+k_1)}{fH_1^2}\left(k_1(1+k_2)A_1+k_2(1+k_1)A_2\right)=0.
\end{aligned} \tag{4.22}$$

Equation for A_2 can be obtained by inchanging (A_1, k_1, H_1) with (A_2, k_2, H_2) respectively. The ansatz consistent with the boundary condition is

$$A^i(u) = \frac{f^{-i\tilde{\omega}(T_0/2T)}}{1 + k_i u} a_i(u), \qquad a_i(u) = p_i(u) + i\tilde{\omega} s_i(u). \tag{4.23}$$

Using above ansatz we obtain equations for p_1, p_2, s_1, s_2. The solution for p_1 is given by,

$$p_1(u) = b_{10}\left(1 + \frac{2k_1}{3}u\right) + b_{20}\frac{k_2(1+k_1)}{3(1+k_2)}u, \tag{4.24}$$

while p_2 can be obtained from above expression by permuting k_1 and k_2. The equations for s_i are more involved and can be written as

$$\begin{aligned} s_1(u)'' + s_1'\left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}}\right) - s_1(u)\frac{k_1}{H_1}\left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}}\right) + \\ -\frac{(1+k_1)u^2}{H_1}\left[\frac{k_1(1+k_2)}{fH_1}s_1(u) - \frac{k_2(1+k_1)}{fH_2}s_2(u)\right] + \\ -2vp_1(u)'\frac{f'}{f} + vp_1(u)\left(\frac{f'}{f}\frac{\mathcal{H}'}{\mathcal{H}} - \frac{f''}{f}\right) = 0, \end{aligned} \tag{4.25}$$

where we used $v = T_0/(2T)$. Equation for s_2 can be obtained by permutation, as usual. Solution is given by,

$$\begin{aligned} s_1(u) =& h_1 u + (h_2 + h_3 u)\arctan\left(\frac{\sqrt{3 - k_1^2 + 2k_2 - k_2^2 + 2k_1(1+k_2)}\,u}{2 + (1 + k_1 + k_2)u}\right) \\ &+ (h_4 + h_5 u)\log\left(\frac{f}{1-u}\right), \end{aligned} \tag{4.26}$$

and similarly for $s_2(u)$. The parameters are as follows:

$$\begin{aligned} h_1 =& \Big((3{+}2k_1{+}2k_2{+}2k_1k_2)\Big[-2(1{+}k_1)k_2(k_1^3{+}k_1^2(1{-}k_2){-}k_1k_2(1{+}k_2){+}k_2^3(1{+}k_2))\Big]b_{20} \\ &+k_1(1+k_2)\Big[4k_1^3 - k_1^2(7k_2-4) + (k_2+1)k_2^2 + k_1k_2(2k_2-7)\Big]b_{10}\Big) \\ &\times\frac{v}{9(1+k_1)(1+k_2)^2[-3+k_1^2-2k_2+k_2^2-2k_1-2k_1k_2]}, \end{aligned} \tag{4.27}$$

$$h_2 = \frac{2h_3}{k_1} + \frac{(1+k_1)}{k_1(1+k_2)}l_3, \tag{4.28}$$

$$\begin{aligned} h_3 =& \Big(2k_1(1+k_2)\Big[9 + k_1^3 + +3k_2 + k_2^2 + k_2^3 + k_1^2(5k_2+13) + k_1(21 + 8k_2 - k_2^2)\Big]b_{10} \\ &-k_2(1{+}k_1)\Big[9{+}k_1^3{+}21k_2{+}13k_2^2{+}k_2^3{+}k_1^2(13{+}5k_2){+}k_1(21{+}20k_2{+}5k_2^2)\Big]b_{20}\Big) \\ &\times\frac{v}{3(1+k_2)(3-k_1^2+2k_2-k_2^2+2k_1+2k_1k_2)^{3/2}}, \end{aligned} \tag{4.29}$$

$$h_4 = (3/2)vb_{10}, \tag{4.30}$$

$$h_5 = k_1 v b_{10} - k_2\frac{1+k_1}{2(1+k_2)}vb_{10}, \tag{4.31}$$

where we have used $v = (T_0/2T)$ and the parameter l_3:

$$l_3 = \Big(2(1+k_1)k_2\left[9+k_1^3-k_1^2(k_2-1)+21k_2+13k_2^2+k_2^3+k_1(3+8k_2+5k_2^2)\right]b_{20}$$
$$-k_1(1+k_2)\left[9+k_1^3+21k_2+13k_2^2+k_2^3+k_1^2(13+5k_2)+21k_1+20k_2+5k_2^2\right]b_{10}\Big)$$
$$\times\frac{v}{3(1+k_1)(3-k_1^2+2k_2-k_2^2+2k_1(1+k_2)^{3/2}} \tag{4.32}$$

The boundary action is given by

$$\mathrm{S}_{\mathrm{boundary}} = \lim_{u\to 0}\frac{N_c^{\frac{3}{2}}\ T_0}{36\sqrt{2}}\int dt d\vec{x}\ \frac{f}{\mathcal{H}}\left[H_1^2 a_1' a_1 + H_2^2 a_2' a_2\right]. \tag{4.33}$$

Now we can follow the same procedure as in the $d = 5$ case and obtain the R-charge conductivity by differentiating twice with respect to b_{10} and b_{20}. The (11) and (12) components of R-charge conductivity turn out to be

$$\lambda_{11} = \frac{N_c^{\frac{3}{2}}\ T_0 v}{36\sqrt{2}}\frac{2\left[4k_1^3(2+k_2)+9(3+3k_2+k_2^2)+k_1^2(45+31k_2+k_2^3)+k_1(63+54k_2+11k_2^2)\right]}{9(1+k_1)(1+k_2)},$$
$$\lambda_{12} = \frac{N_c^{\frac{3}{2}}\ T_0 v}{36\sqrt{2}}\frac{2}{9(1+k_1)^2(1+k_2)^2}\Big[k_1^4 k_2(2+k_2)+k_2(9+9k_2+2k_2^3)+k_1^3(2+18k_2+13k_2^2+2k_2^3)+$$
$$+k_1^2(9+54k_2+48k_2^2+13k_2^3+k_2^4)+k_1(9+54k_2+54k_2^2+18k_2^3+2k_2^4)\Big]. \tag{4.34}$$

The (22) component can be ontained by interchanging k_1, b_{10} with k_2, b_{20} respectively. Once again since all the components of susceptibility diverges near critical point, while conductivity remains finite, as evident from the above equations, the diffusion coefficients will vanish.

4.3 Seven dimensional black hole with two charges

The seven dimensional case corresponds to the M5 brane. Since the transverse space is five dimensional only two U(1) charges are possible. The equations for the perturbation in gauge field is given by

$$A_1'' + \left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}} + 2\frac{H_1'}{H_1} - \frac{1}{u}\right)A_1' - \frac{4(1+k_1)u^3}{fH_1^2}\Big[k_1(1+k_2)A_1 + k_2(1+k_1)A_2\Big] = 0. \tag{4.35}$$

The other equation can be obtained by interchanging A_1, k_1 with A_2 and k_2 respectively. The ansatz appropriate to the boundary condition is

$$A_1(u) = \frac{f^{-i\tilde{\omega}v}}{H_1}\left(p_1(u) + i\tilde{\omega}s_1(u)\right). \tag{4.36}$$

The explicit form of the solution to $s_i(u)$'s are obtained but they are not written here as they are quite complicated. But the conclusion remains the same i.e. as we approach critical line diffusion constant goes to zero. We explain in the appendix (see last section) why two-charge case for M5 brane is complicated than the same for D3 and M2 brane.

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5 Conclusion

In this paper we have investigated some of the features of equilibrium and non-equilibrium thermodynamic properties of R-charged AdS black hole in its full generality near the critical point. Furthermore, by making use of the gauge-gravity correspondence, we tried to extract some properties of strongly coupled boundary gauge theories at finite temperature. More specifically, we analyzed five, four and seven dimensional black holes with multiple charges. This led to the construction of Bragg-Williams potentials associated with these black holes. We discussed how Bragg-Williams potential allow us to make a proposal for the gauge theory effective potential (above the critical temperature). This effective potential is usefull in determining the equilibrium properties of the strongly coupled gauge theories in the presence of non-zero chemical potential. Subsequently, we analysed some of the non-equilibrium properties of the black holes and its duals. Though, for single R-charged black holes, thermal properties were discussed in [1] and [7], we extended these results in several ways. Besides providing a general framework to solve the perturbed gravity equations, we determined the conductivities and diffusion coefficients for multiply charged black holes.

Our work may be extended futher by considereing rotating R-charged black holes. They were consuructed, for example, in [21]. Another question that is of crucial importance is to understand the fate of these R-charged black holes below the critical temperature. As we saw, these holes become unstable. However, the new phase to which it crosses over is, to our knowledge, not yet clearly understood.

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A General solution to linearized equation for perturbation

In this appendix we briefly sketch the structure of the solutions and the general method that have been used to solve linearized equations for the perturbations in the gauge field. The equations satisfied by p_i and s_i in each of the cases, i.e. for five four and seven dimensional black hole can be written in the following general form:

$$\begin{gathered} p_i''(u) + \left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}} + \frac{c}{u}\right) p_i'(u) + \\ -\frac{k_i}{H_i}\left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}}\right) p_i(u) - u^n \frac{1+k_i}{fH_i} \prod_{i=1}^{m}(1+k_i)\left(\sum_{i=1}^{m} \frac{k_i}{(1+k_i)H_i} p_i\right) = 0, \quad i = 1,..,m. \end{gathered} \tag{A.1}$$

$$s_i''(u) + \left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}} + \frac{c}{u}\right) s_i'(u) - \frac{T_0}{T}\frac{f'}{f} p_i'(u) + \frac{T_0}{2T}\left(\frac{f'\mathcal{H}'}{f\mathcal{H}} - \frac{f''}{f}\right) p_i +$$
$$- \frac{k_i}{H_i}\left(\frac{f'}{f} - \frac{\mathcal{H}'}{\mathcal{H}}\right) s_i(u) - u^n \frac{b(1+k_i)}{fH_i} \prod_{j=1}^{m}(1+k_j)\left(\sum_{i=1}^{m} \frac{k_i}{(1+k_i)H_i} p_i\right) = 0, \quad i = 1,..,m, \tag{A.2}$$

where m is the number of R-charges. The parameters c, n, b and m have following values for the different cases:

- Five dimensional black hole: $c = 0$, $n = 1$, $b = 1$, $m = 1, 2, 3$.
- Four dimensional black hole: $c = 0$, $n = 2$, $b = 1$, $m = 1, 2, 3, 4$.
- Seven dimensional black hole: $c = -1$, $n = 3$, $b = 4$, $m = 1, 2$.

The solutions should be regular at horizon i.e. at $u = 1$. As already mentioned in the main text, without any loss of generality we can also impose the condition that at $u = 0$ the term linear in frequency should vanish.

Solution for p_i's can be written generically as

$$p_i(u) = b_{i0} + c_{i0}\ u^l, \tag{A.3}$$

where c_{i0} depends on all the k_i's and b_{i0}'s. Clearly b_{i0} is the constant representing boundary value of $A_i(u)$. For five and four dimensional black hole $l = 1$ while for seven dimensional black hole $l = 2$.

Now we turn to solution for s_i's. The function $f(u)$ in the metric generically can be written as,

$$f = (1-u)F(u), \tag{A.4}$$

where $F(u)$ is a polynomial in u of order ν, and so has ν number of zeroes given by $u = u_j, j = 1, 2, .., \nu$. For four and five dimensional black hole $\nu = m - 1$ while for seven dimensional blackhole $\nu = m + 2$

Then most general solution for s_i can be written as

$$s_i = P_i(u) + \sum_{j=1}^{\nu} Q_{ij}(u) \log(1 - u/u_j), \tag{A.5}$$

where $P_i(u)$ and $Q_{ij}(u)$ are two sets of functions. In order to ensure $s_i(0) = 0$, we have $P_i(0) = 0$. We require our solution to be regular at the horizon so $\log(1-u)$ term is absent. This is the genral form representing the solution for various cases as discussed in main text as well as in appendix. For each particular case we need to determine the functions $P_i(u)$ and $Q_{ij}(u)$.

B Five dimensional black hole with three charges

In the case of five dimensional black hole with three charges the values of the parameters are $c = 0$, $n = 1$, $b = 1$, $m = 3$ and $l = 1$. Since $f = (1-u)(1+(1+k_1+k_2+k_3)u - k_1k_2k_3u^2)$ we get

$$F(u) = (1 + (1 + k_1 + k_2 + k_3)u - k_1k_2k_3u^2), \tag{B.1}$$

which is a quadratic polynomial implying $\nu = 2$. It is helpful for calculation if we write the second term in (A.5) as

$$Q_{i1}\log(1-u/u_1)+Q_{i2}\log(1-u/u_2)=\frac{Q_{i1}+Q_{i2}}{2}\log(F(u))+\frac{Q_{i1}-Q_{i2}}{2}\log\left(\frac{u_2(u_1-u)}{u_1(u_2-u)}\right). \tag{B.2}$$

Using $l = 1$ we write down $p_i(u)$ from (A.3) as

$$p_i(u) = b_{i0} + c_{i0}u. \tag{B.3}$$

c_{10} turns out to be

$$c_{10} = \frac{1}{2}\left[k_1 b_{10} - (1+k_1)\left(\frac{k_2}{1+k_2}b_{20} + \frac{k_3}{1+k_3}b_{30}\right)\right], \tag{B.4}$$

Other components of c_{i0} can be obtained by cyclic permutation of (123).

Expressions of $s_i(u)$ are as follows:

$$\begin{aligned} s_1(u) =& h_1 u + (h_2 + h_3 u)\log\left(\frac{2+(1+k_1+k_2+k_3+\sqrt{\triangle})u}{2+(1+k_1+k_2+k_3-\sqrt{\triangle})u}\right) \\ &+ (h_4 + h_5 u)\log[1+(1+k_1+k_2+k_3)u - k_1k_2k_3u^2], \end{aligned} \tag{B.5}$$

while

$$\triangle = (1+k_1+k_2+k_3)^2 + 4k_1k_2k_3. \tag{B.6}$$

h_i for $i = 1,2,3,4,5$ are constants, which depend on k_1, k_2 and k_3 and boundary values of the gauge fields, namely, b_{10}, b_{20}, b_{30}. Introducing some additional parameters, h_1 can be written succintly as

$$h_1 = \frac{(2+k_1+k_2+k_3-k_1k_2k_3)v}{4(1+k_2)(1+k_3)\triangle}\left[\frac{k_1J_1}{1+k_1}b_{10} - \frac{k_2J_2}{1+k_2}b_{20} - \frac{k_3J_3}{1+k_3}b_{30}\right]. \tag{B.7}$$

These new parameters J_1, J_2 and J_3 are fourth order polynomials in k_i's and are given as

$$\begin{aligned} J_1 =& k_1^4 - k_1^3(-2+k_2+k_3) - k_1^2(k_2-k_3)^2 + 4(k_2+k_3)(k_1^2+k_2k_3) + k_1^2 - 4k_2k_3 \\ &+ k_1(k_2^3 - 2k_2^2 - 5k_3k_2^2 - 3k_3 - 2k_3^2 + k_3^3 - 3k_2 - 5k_2k_3^2), \\ J_2 =& [k_1^3 + k_2^3 - k_3^3 - k_1^2k_2 - k_1k_2^2 + 3k_1^2k_3 + 3k_2k_3^2 + 10k_1k_2k_3] \\ &+ [2(-k_1^2 + k_2^2 + 6k_3^2) + 4(k_1+k_2)k_3] + [k_1 + k_2 + 7k_3], \\ J_3 =& [k_1^3 - k_2^3 + k_3^3 - 3k_1^2k_2 + 3k_1k_2^2 + 3k_2^2k_3 - 3k_2k_3^2 - k_1^2k_3 - k_1k_3^2 + 10k_1k_2k_3] \\ &+ [2k_1^2 + 6k_2^2 + 2k_3^2 + 4k_2k_3 + 4k_1k_2] + [k_3 + 7k_2 + k_1]. \end{aligned} \tag{B.8}$$

Similarly h_2 is given by,

$$\begin{aligned} h_2 = -\frac{1+k_1}{2\sqrt{\triangle}}\Bigg[& 2k_1(1+k_2)(2+k_1+k_2+k_3-k_1k_2k_3)\left[\frac{k_2L_2}{1+k_2}b_{20} + \frac{k_3L_3}{1+k_3}b_{30}\right] \\ &+ (1+k_2)(1+k_3)L_1b_{10}\Bigg]v, \end{aligned} \tag{B.9}$$

where L_i's are

$$\begin{aligned}
L_1 =& -1+k_2^3+k_3+3k_3^2+k_3^3+3k_2^2+3k_2^2k_3-k_1^3+2k_1^3k_2k_3+k_2+6k_2k_3+3k_2k_3^2-3k_1+k_1k_3^2\\
&+16k_1k_2k_3+6k_1^2k_2k_3^2+k_1k_2^2+6k_1k_2^2k_3+k_1^2[-3-k_3+2k_2^2k_3-k_2+8k_2k_3+2k_2k_3^2],\\
L_2 =&(1+k_1+k_2-k_3),\\
L_3 =&(1+k_1-k_2+k_3).
\end{aligned} \tag{B.10}$$

h_3 is given by

$$h_3 = \frac{k_1}{2}h_2 - \frac{v}{2(1+k_2)(1+k_3)\triangle^{3/2}}(k_2M + k_3N), \tag{B.11}$$

where v depends on k_i's in the following manner

$$v = \frac{\sqrt{(1+k_1)(1+k_2)(1+k_3)}}{2+k_1+k_2+k_3-k_1k_2k_3}. \tag{B.12}$$

M is given as follows.

$$\begin{aligned}
M =& -2k_2(1+k_2)(2+k_1+k_2+k_3-k_1k_2k_3)\Big[k_1(1+k_3)(1+k_1+k_2-k_3)b_{10}\\
&\qquad + k_3(1+k_1)(1-k_1+k_2+k_3)b_{30}\Big]+\\
&+(1+k_1)(1+k_3)\Big[-1+k_1^3-k_2^3+k_3+3k_3^2+k_3^3-3k_2^2-k_3k_2^2\\
&\quad + k_1^2(3+k_2+3k_3+6k_2k_3+2k_2^2k_3)+k_2(-3+k_3^2)+\\
&\quad + k_1(1+6k_3+2k_2^2k_3+3k_3^2+16k_2k_3+6k_2k_3^2-k_2^2+8k_3k_2^2+2k_2^2k_3^2)\Big]b_{20}.
\end{aligned} \tag{B.13}$$

N can be obtained by interchanging k_2 and k_3 in the above expression. The rest of the parameters h_4 and h_5 are

$$\begin{aligned}
h_4 &= \frac{3v}{2}b_{10}\\
h_5 &= \frac{3v}{4}(1+k_1)\left[\frac{k_1}{1+k_1}b_{10} - \frac{k_2}{1+k_2}b_{20} - \frac{k_3}{1+k_3}b_{30}\right]
\end{aligned} \tag{B.14}$$

Evaluating the boundary action

$$S_{\text{boundary}} = \frac{N_c^{(3/2)}T_0}{36\sqrt{2}} \lim_{u\to 0} \int dt d\vec{x}\, \frac{f}{\mathcal{H}}\left(H_1^2a_1'a_1 + H_2^2a_2a_2' + H_3^2a_3a_3'\right), \tag{B.15}$$

one obtains the expression of the boundary action upto quadratic in b_{10}, b_{20},b_{30}. This expression is too long and so we are not writing it explicitly. The R-charge conductivity can be obtained by differentiating boundary action twice and are given by

$$\begin{aligned}
\lambda_{11} =& \frac{N_c^{(3/2)}T_0}{36\sqrt{2}}\Big[k_1^2(-1+k_2k_3)+k_1^2(k_2^2k-3-2(5+3k_3)+k_2(-6+k_3^2))\\
&-4(2+2k_3+k_3^2+k_2^2(1+k_3)+k_2(2+2k_3+k_3^2)-k_1(16+14k_3+5k_3^2+k_2^2(5+4k_3)+\\
&+2k_2(7+5k_3+2k_3^2)\Big]\frac{v}{2(1+k_1)(1+k_2)(1+k_3)},
\end{aligned} \tag{B.16}$$

$$\lambda_{12} = \frac{N_c^{(3/2)} T_0}{36\sqrt{2}} \; \frac{1}{4}\Big[k_1(4+k_1+k_2+k_3)(-2-k_2-k_3+k_1(-1+k_2k_3)(1+k_3))\Big] \\ + \left(\frac{v}{(1+k_1)^2} + \frac{v}{(1+k_2)^2}\right), \tag{B.17}$$

Other components of the conductivity matrix can be obtained from above two components by cyclically permuting k_1, k_2 and k_3. The diffusion coefficients can be obtained from the relation $D_{ij} = \sum_{k=1}^{3} \lambda_{ik}\chi_{kj}^{-1}$, where χ is the specific heat. Since all the components of specific heat diverges the diffusion coefficients turn out to be zero, as expected.

C Four dimensional black hole with three charges

For four dimensional black hole with three charges we have $c = 0, n = 2, b = 1$, $m = 3$ and $l = 1$. From $f(u) = (1-u)[1+(1+k_1+k_2+k_3)u+(1+k_1+k_2+k_3+k_1k_2+k_2k_3+k_1k_3)u^2)$ we get

$$F(u) = 1 + (1+k_1+k_2+k_3)u + (1+k_1+k_2+k_3+k_1k_2+k_2k_3+k_1k_3)u^2), \tag{C.1}$$

which is quadratic. The solutions are given by

$$p_1(u) = b_{10}\left(1+\frac{2k_1}{3}\right) + \frac{(1+k_1)}{3(1+k_2)(1+k_3)}\Big[k_2(1+k_3)b_{20} + k_3(1+k_2)b_{30}\Big], \tag{C.2}$$

while other p_i's can be obtained from above expression by permuting (123) cyclically.

Using (A.5) $s_i(u)$'s is given by,

$$\begin{aligned} s_1(u) &= h_1u + (h_2+h_3u)\triangle(u) + (h_4+h_5u)\log F(u), \\ s_2(u) &= l_1u + (l_2+l_3u)\triangle(u) + (l_4+l_5u)\log F(u), \\ s_3(u) &= m_1u + (m_2+m_3u)\triangle(u) + (m_4+m_5u)\log F(u), \\ \triangle(u) &= \arctan\left[\frac{\sqrt{3-k_1^2-k_2^2+2k_3-k_3^2+2k_2(1+k_3)+2k_1(1+k_2+k_3)}}{2+(1+k_1+k_2+k_3)u}u\right]. \end{aligned} \tag{C.3}$$

Here once again we have combined two logarithms as in (B.2). The values of the parameters are as follows:

$$\begin{aligned} h_3 &= \frac{\Big[k_3 - 2k_1(1+k_2)(1+k_3)\Big]h_2 + (1+k_1)k_2(1+k_3)l_2 + (1+k_1)k_2k_3m_2}{3(1+k_2)(1+k_3)}v \\ h_4 &= \frac{3v}{2}b_{10} \\ h_5 &= \frac{2k_1(1+k_2)(1+k_3)b_{10} + (1+k_1)[k_2(1+k_3)b_{30} + k_3(1+k_2)b_{30}]}{2(1+k_2)(1+k_3)}v, \\ h_2 &= \Bigg(4(1+k_1)\Big[3+2k_1+2k_2+2k_3+2k_1k_2+2k_2k_3+2k_3k_1\Big] \\ &\quad + \Big[k_2(1+k_3)(1+k_1+k_2-k_3)b_{20} + k_3(1+k_2)(1+k_1-k_2+k_3)b_{30}\Big] \end{aligned} \tag{C.4}$$

$$
\begin{aligned}
&-\Big[(1+k_2)(1+k_3)[4+k_1^4+k_2^4+k_3^4+4k_3+6k_3^2-2k_2^3(1+k_3)+2k_1^3(7+3k_2+3k_3)+\\
&+2k_2^2(-3-k_3+k_3^2)-2k_2(-3-3k_3+k_3^2+k_3^3)+2k_1^2(17+k_2^2+11k_3+k_3^2+11k_2+3k_2k_3)\\
&-2k_1(-15+k_2^3-11k_3+k_3^2+k_3^3+k_2^2+k_2^2k_3-11k_2-4k_2k_3+k_2k_3^2)\Big]\Big)v\\
&\times\Big((1+k_2)(1+k_3)\Big[k_1^3+k_2^3-k_2^2(1+k_3)+(k_3-3)(1+k_3)^2-k_1^2(1+k_2+k_3)\\
&-k_2(5+6k_3+k_3^2)-k_1(5+k_2^2+6k_3+k_3^2+6k_2+6k_2k_3)\Big]\\
&\times\sqrt{(3-k_1^2-k_2^2-k_3^2+2k_1+2k_2+2k_3+2k_1k_2+2k_2k_3+2k_3k_1)}\Big)^{-1}, \qquad \text{(C.5)}
\end{aligned}
$$

where we have used $v = T_0/(2T)$. The components of R-charge conductivity turn out to be

$$
\begin{aligned}
\lambda_{11} =&2\Big(4k_1^4\Big[2+k_2+k_3\Big]+k_1^3\Big[53+5k_2^2+43k_3+5k_3^2+k_2(43+23k_3)\Big]\\
&+k_1^2\Big[108+k_2^3+130k_3+43k_3^2+k_3^3+k_2^2(43+26k_3)+2k_2(65+60k_3+13k_3^2)\Big]\\
&+9\Big[3+6k_3+4k_3^2+k_3^3+k_2^3(1+k_3)+k_2^2(4+6k_3+3k_3^2)+k_2(6+10k_3+6k_3^2+k_3^3)\Big]\\
&+k_1\Big[90+144k_3+74k_3^2+11k_3^3+k_2^3(11+10k_3)\\
&\quad+k_2^2(74+82k_3+29k_3^2)+k_2(144+187k_3+82k_3^2+10k_3^3)\Big]\Big)\\
&\times\left(\frac{v}{9(1+k_1)(1+k_2)(1+k_3)(1+k_1+k_2+k_3)}\right), \qquad \text{(C.6)}\\
\lambda_{12} =&-\frac{\Big[k_2+k_1^2k_2+k_1(1+4k_2+k_2^2)\Big]v}{9(1+k_1)^2(1+k_2)^2(1+k_3)(1+k_1+k_2+k_3)}\Big(2k_2^3\Big[2+k_3\Big]+2k_1^3\Big[2+k_2+k_3\Big]+\\
&+k_2^3\Big[22+14k_3+k_3^2\Big]-2\Big[-9-9k_3+k_3^2+k_3^3\Big]-k_2\Big[-36-26k_3-4k_3^2+k_3^3\Big]+\\
&+k_2^2\Big[22+4k_2^2+14k_3+k_3^2+k_2(20+7k_3)\Big]-2\Big[-9-9k_3+k_3^2+k_3^3\Big]-\\
&+k_2\Big[-36-26k_3+4k_3^2+k_3^3\Big]+k_1^2\Big[22+4k_2^2+14k_3+k_3^2+k_2(20+7k_3)\Big]\\
&+k_1\Big[36+2k_2^3+26k_3-4k_3^2-k_3^3+k_2^2(20+7k_3)+k_2(50+24k_3-8k_3^2)\Big]\Big) \qquad \text{(C.7)}
\end{aligned}
$$

while other components can be obtained by cyclically permuting k_1, k_2 and k_3.

D Four dimensional black hole with four charges

For four dimensional black hole with four charges we have $c = 0, n = 2, b = 1$, $m = 4$ and $l = 1$. $f(u)$ is given by

$$
\begin{aligned}
f =&(1-u)[1+(1+k_1+k_2+k_3+k_4)u+\\
&+(1+k_1+k_2+k_3+k_4+k_1k_2+k_1k_3+k_1k_4+k_2k_3+k_2k_4+k_3k_4)u^2-k_1k_2k_3k_4u^3], \qquad \text{(D.1)}
\end{aligned}
$$

and so we have

$$F(u) = 1 + (1 + k_1 + k_2 + k_3 + k_4)u + \\ + (1{+}k_1{+}k_2{+}k_3{+}k_4{+}k_1k_2{+}k_1k_3{+}k_1k_4{+}k_2k_3{+}k_2k_4{+}k_3k_4)u^2 - k_1k_2k_3k_4u^3. \tag{D.2}$$

which is a cubic polynomial.

From (A.3) we can write

$$p_i(u) = b_{i0} + c_{i0}u, \tag{D.3}$$

where b_{i0} are constant parameters, c_{10} is given by

$$c_{10} = \frac{2k_1}{3}b_{10} - \frac{1+k_1}{3}\left[\frac{k_2}{1+k_2}b_{20} - \frac{k_3}{1+k_3}b_{30} - \frac{k_4}{1+k_4}b_{40}\right] \tag{D.4}$$

and other c_{i0} can be obtained by permuting (1234) in the above expression. From (A.5) we obtain

$$s_1(u) = h_1u + (h_2 + h_3u)\log(1 - u/u_1) + (h_4 + h_5u)\log(1 - u/u_2) \\ + (h_6 + h_7u)\log(1 - u/u_3) \tag{D.5}$$

Since $F(u)$ is a polynomial of degree 3 the zeroes of $F(u)$, $u_i, i = 1, 2, 3$ have complicated expressions. Because of the complicated, which in turn makes explicit expressions of parameters h_1 to h_7 too long to include here. Once again proceeding in a similar way we obtain the R-charge conductivities in terms of k_i's.

E Seven dimensional black hole with two charges

In the case of seven dimensional black hole with two charges we have $c = -1$, $n = 3$, $b = 4$ and $m = 3$.

$$f = (1 - u)[1 + u + (1 + k_1 + k_2 + k_1k_2)u^2 - k_1k_2u^3]. \tag{E.1}$$

So $F(u)$ is given by

$$F(u) = 1 + u + (1 + k_1 + k_2 + k_1k_2)u^2 - k_1k_2u^3, \tag{E.2}$$

which is, once again, cubic.

Since $l = 2$ from (A.3) we get

$$p_i(u) = b_{i0} + c_{i0}u^2. \tag{E.3}$$

From (A.5) we get

$$s_1(u) = h_1u + h_2u^2 + (h_3 + h_4u^2)\log(1 - u/u_1) + (g_5 + g_6u^2)[\log(1 - u/u_2) \\ + (g_7 + g_8u^2)\log(1 - u/u_3). \tag{E.4}$$

$F(u)$ being of order 3, expressions of zeroes u_i's are more involved which makes the expressions of h_1 to h_8 too long and complicated to write down explicitly. We have analyzed this case and obtain similar conclusion that as critical surfaces are approached diffusion coefficients vanish.

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